



Chapter 3: Shortcuts to Differentiation

Section 3.7 - Implicit Functions
Section 3.8 - Hyperbolic Functions
Section 3.9 - Linear Approximation

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3.7 - Implicit Functions

A function $y = f(x)$ is called **explicit** since it is written as y in terms of x . An equation such as

$$f(x, y) = c, \text{ where } c \text{ is a constant}$$

is called an **implicit** function.

An example of an implicit function is the circle $x^2 + y^2 = 1$. How do we obtain an expression for $\frac{dy}{dx}$?

We simply take the derivative of the whole expression with respect to x :

$$\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} 1$$

The right side goes to zero, and we use the chain rule on the left:

$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

Solving for dy/dx , we get

$$\frac{dy}{dx} = \frac{-x}{y}.$$



3.7 - Implicit Functions

Ex: Differentiate each of the following:

a. $x \ln(y) = 1$

b. $x^2 + ye^y = 2$

c. $\cos(x) + \sin(y) = 0$

d. $x\sqrt{1+y} + y\sqrt{1+2x} = 2x$

e. $x \sin(y) + \cos(2y) = \cos(y)$

f. $xy = \cot(xy)$

Ex: Find the equation of the tangent line to the curve with equation $x^5 - xy^2 + y^3 = 4$ at the point $(1, 2)$.

Ex: Prove the curves $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ are orthogonal. (Show the tangent lines are perpendicular at every point of intersection)



3.8 - Hyperbolic Functions

We define $\cosh x$ and $\sinh x$ by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

$\tanh x$ is defined as expected:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Many properties of trig functions hold also for hyperbolic functions. Calculate each of the following:

a. $\cosh 0 =$

b. $\sinh 0 =$

c. $\cosh(-x) =$

d. $\sinh(-x) =$

e. $(\cosh x)^2 - (\sinh x)^2 =$



3.8 - Derivatives of Hyperbolic Functions

Find each of the following:

$$\frac{d}{dx} \sinh x =$$

$$\frac{d}{dx} \cosh x =$$

$$\frac{d}{dx} \tanh x =$$

Ex: Simplify $f(x) = \sinh(\ln(x))$.

Ex: Find each of the following derivatives

a. $y = \cosh(t) \sinh(2t)$

b. $f(\theta) = \ln(\cosh(2 + \theta))$

c. $g(x) = \sinh(\sinh(x^2))$



3.9 - Linear Approximation

Look at any differentiable function on a small interval. What does it look like? For example, what does $\sin(x)$ look like for $-.1 < x < .1$?

Suppose we want to approximate a differentiable function, $f(x)$, at a point $x = a$ by a linear function. How can we do this? What equation do we get?

The Tangent Line Approximation

Suppose f is differentiable at a . Then, for values of x near a , the tangent line approximation to $f(x)$ is given by

$$f(x) \approx f(a) + f'(a)(x - a).$$

This expression $f(a) + f'(a)(x - a)$ is called the local linearization of f near $x = a$.

The error of the linear approximation is

$$E(x) \approx \frac{f''(a)}{2}(x - a)^2$$



3.9 - Linear Approximation (An Aside)

What if a linear approximation isn't good enough? Can I use a quadratic?

$$f(x) \approx f(a) + f'(a)(x - a) + f''(a)(x - a)^2.$$

What if a quadratic approximation isn't good enough?

Taylor Series Approximation

Suppose f is a function with infinitely many derivatives. For any point $x = a$, we have

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(a)(x - a)^i$$

Notice this is $=$ and not \approx . The Taylor series is one extremely important example of how derivatives can be used in practice.



3.9 - Linear Approximation

Ex: Find the linear approximation to

$$f(x) = \frac{1}{1+x^2} \text{ at } x = 1.$$

Ex: What is the local linearization of e^{-x} near $x = 1$?

Ex: The acceleration due to gravity, g , is given by

$$g = \frac{GM}{r^2}$$

where M is the mass of the earth, r is the distance from the center of the earth, and G is the universal gravitational constant. Show that when r changes by Δr , the change in the acceleration due to gravity, Δg is given by

$$\Delta g \approx -2g \frac{\Delta r}{r}.$$



Chapter 3 Review Questions

Ex: Are the following statements true or false? Explain.

- a. The 100^{th} derivative of $\sinh x$ is $\cosh x$.
- b. If $f(x)$ is defined for all x , then $f'(x)$ is defined for all x .
- c. If $f(x)$ is increasing then $f'(x)$ is increasing.
- d. $(fg)'(x)$ is never equal to $f'(x)g'(x)$.
- e. The only functions whose fourth derivatives are equal to $\cos t$ are of the form $\cos t + C$ where C is any constant.
- f. If f and g are concave up for all x then $f(x) + g(x)$ is concave up for all x .
- g. If f and g are concave up for all x then $f(x)g(x)$ is concave up for all x .
- h. If f and g are concave up for all x then $f(x) - g(x)$ cannot be concave up for all x .
- i. If f and g are concave up for all x then $f(g(x))$ is concave up for all x .