Chapter 4: Using the Derivatve

Section 4.1 Using First and Second Derivatives

Section 4.2 Families of Curves

Section 4.3 Optimization

Math 124, Section 023, Fall 2007

Instructor: Paul Dostert

4.1 - Using 1^{st} and 2^{nd} Derivatives

Recall the properties of the 1^{st} and 2^{nd} derivative of a function f.

- If $f' > 0$ on an interval then f is increasing on that interval.
- If $f' < 0$ on an interval then f is decreasing on that interval.
- If $f'' > 0$ on an interval then f is concave up on that interval.
- If $f'' < 0$ on an interval then f is concave down on that interval.

Ex: Draw f using the properties of the derivatives for $f(x) = 4x^3 - 9x^2 + 6x + 2$.

4.1 - Local Extrema

- f has a **local minimum** at $x = a$ if $f(a) \leq f(x)$ for x near a.
- f has a **local maximum** at $x = a$ if $f(x) \leq f(a)$ for x near a.

For any function f, a point $x = a$ where $f'(a) = 0$ or $f'(a)$ is undefined is called a **critical point** of the function. The critical value of f is the value, $f(a)$, at a critical point $x = a$.

Local Extrema and Critical Points
Suppose f is defined on an interval and has a local min/max at a point $x = a$, which is not an endpoint of the interval. If f is differentiable at $x = a$ then $f'(a) = 0$.

Note: A critical point is not necessarily a local min/max.

4.1 - Testing Local Extrema

The First Derivative Test Suppose $x = a$ is a critical point of f.

- If $f'(x)$ changes from negative to positive at $x = a$, then f has a local min at $x = a$.
- If $f'(x)$ changes from positive to negative at $x = a$, then f has a local max at $x = a$.

The Second Derivative Test Suppose $x = a$ is a critical point of f.

- If $f''(a) > 0$ then f has a local min at $x = a$.
- If $f''(a) < 0$ then f has a local max at $x = a$.
- If $f''(a) = 0$ then we known nothing about the local extrema.

Ex: Find and classify the local extrema of each of the following

a. $f(t) = t^3 + 6t^2 + 3t - 1$.

b. $g(z) = (z^2 - z)^{(1/3)}$

c. $h(x) = xe^{2x}$

4.1 - Inflection Points

If the graph of a function changes concavity at $x = a$ then a is called an **inflection point** of f. This implies f'' changes sign at a and f' has a local minimum or maximum at $x = a$.

Note: At inflection points, f'' is zero or undefined. If $f''(a) = 0$ or $f''(a)$ is undefined, then a is not necessarily an inflection point.

Ex: Find the critical points and inflection points for each of the following. Classify the critical points.

a. $f(x) = e^{-x^2}$

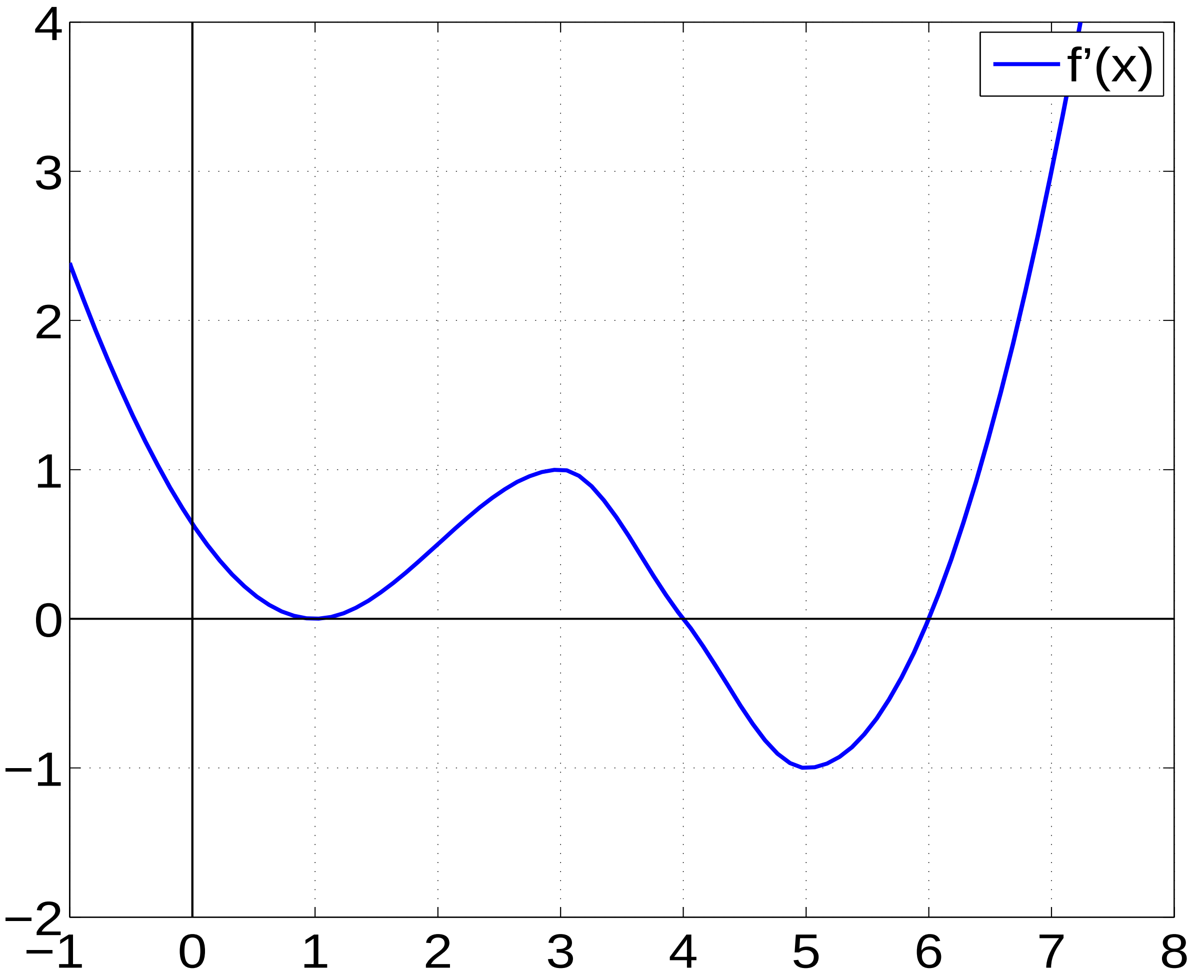
b. $f(x) = x^4 - 6x^2$

c. $f(x) = x(1 - x)^4$

4.1 - Critical and Inflection Points

Ex: Find constants a and b so that $f(x) = x^2 + ax + b$ has a minimum at the point $(3, 5)$.

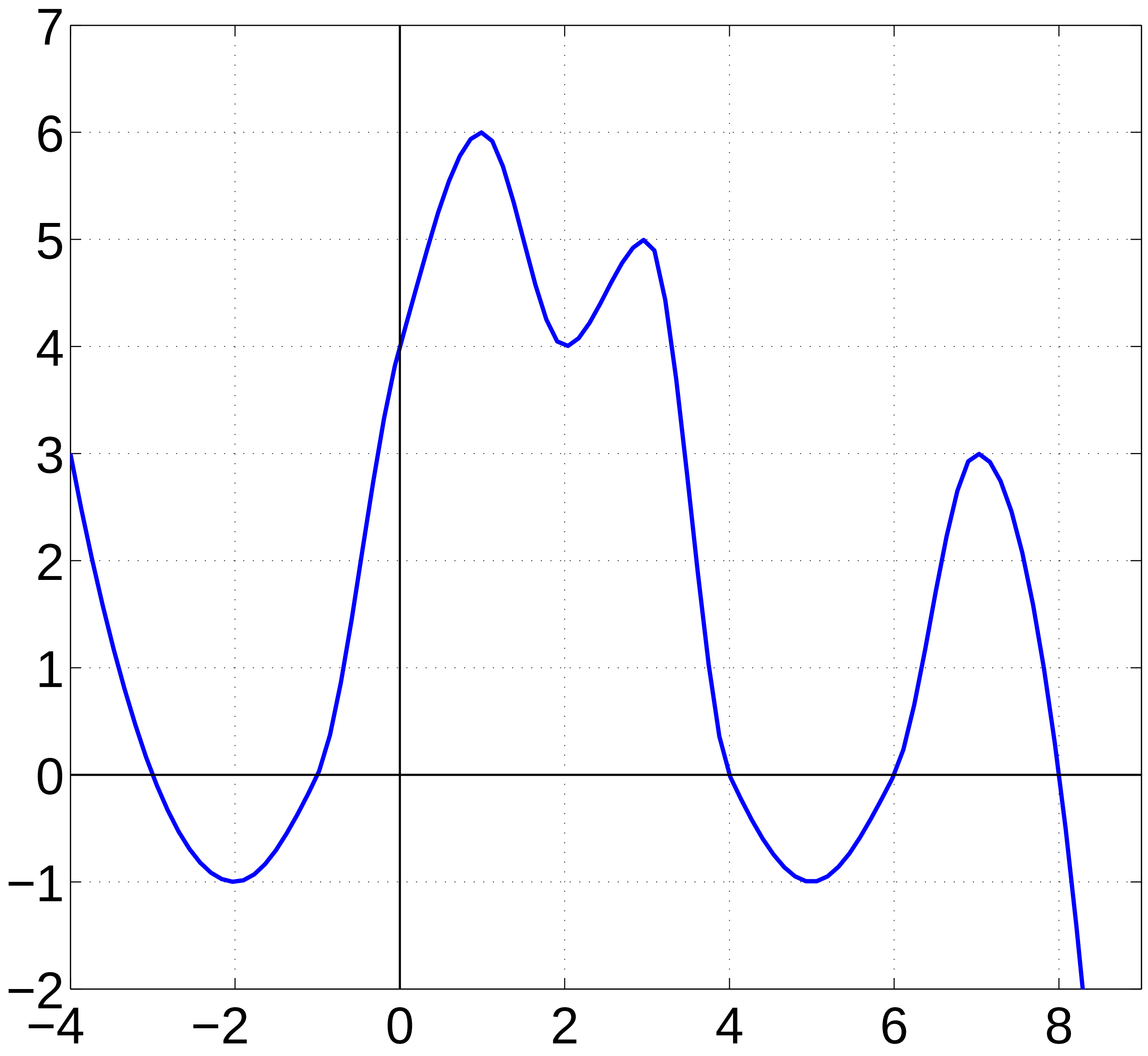
Ex: The graph of $f'(x)$ is given by:



a. What are the critical pts of $f(x)$?

b. Identify each critical pt as a local min, local max, or neither.

4.1 - Critical and Inflection Points



a. If the above graph is of $f(x)$ then many inflection points does $f(x)$ have?

b. If the above graph is of $f'(x)$ then how many inflection points does $f(x)$ have?

c. If the above graph is of $f''(x)$ then how many inflection points does $f(x)$ have?

4.2 - Families of Curves

The goal of this section is to reproduce the equation of a curve with some given properties. We will do this simply by examples.

Ex: Find an equation for each of the following

a. A downward opening parabola with a vertex at $(1, 1)$.

b. The top half of a circle centered at $(-2, 3)$ with a radius of 4

c. A rational function of the form

$$y = ax/(x + b)$$

with a vertical asymptote at $x = 6$ and a horizonal asymptote of $y = 4$.

4.2 - Families of Curves

Ex: Find an equation for each of the following

a. A function of the form $y = A\sin(Bx) + C$ with a maximum at $(3, 9)$ and a minimum $(11, 2)$.

b. A function of the form $y = be^{-(x-a)^2/2}$ with its maximum at the point $(1, 4)$.

c. A quartic polynomial, symmetric about the y-axis, with local maxima at $(-2, 2)$ and $(2, 2)$ and a y-intercept of 1.

Ex: Consider $p(x) = x^3 - ax$. What are the local maximum and minimum?

Ex: Consider $y = x - k\sqrt{x}$ where $k > 0$ and $x \geq 0$. Show $f(x)$ has a local minimum at a point whose x-coordinate is $1/4$ of the way between its x-intercepts.

4.2 - Families of Curves

Ex: Consider $y = axe^{-bx}$ for $a, b > 0$. Find local min/max and inflection points.

Ex: For positive a and b the potential energy of a particle is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$$

for $x > 0$. Find the intercepts, asymptotes, and local extrema.

Ex: An organism has size W at time t. For positive constants A, b, and c

$$W = Ae^{-e^{b-ct}}, \ t \geq 0.$$

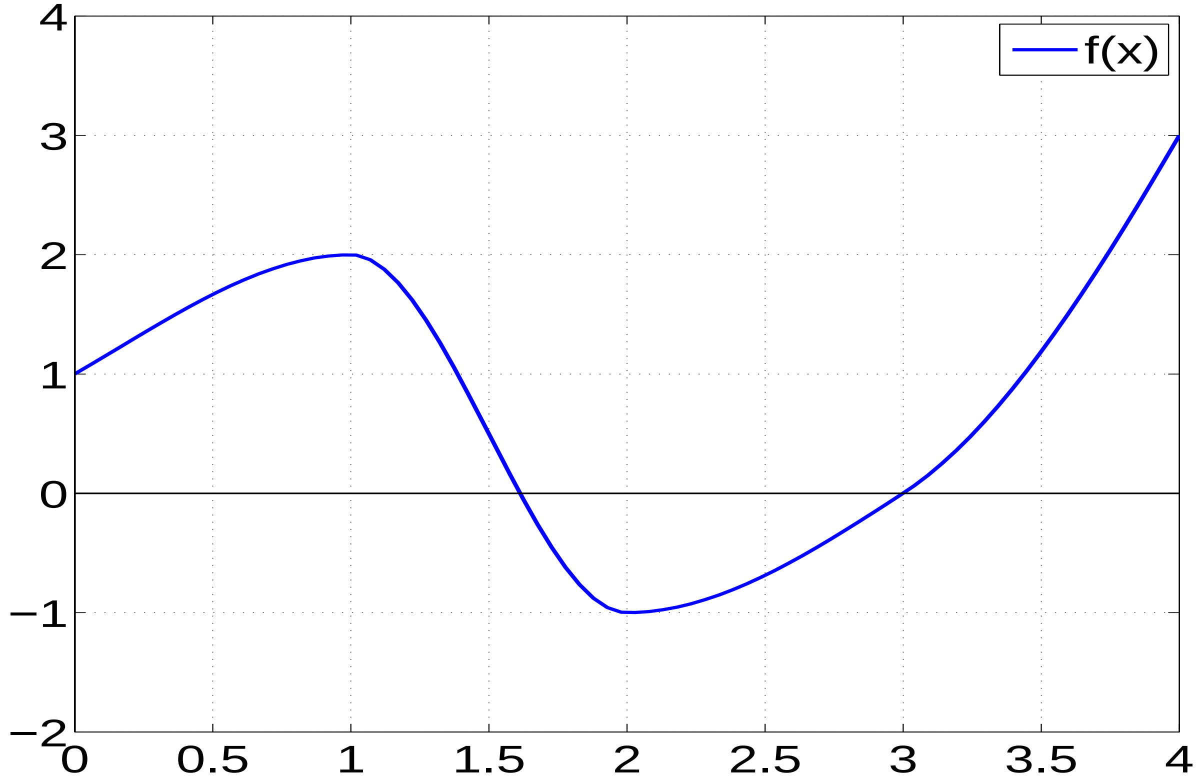
Find the intercepts, asymptotes, critical points, and inflection points.

4.3 - Optimization

The derivative helps us find local extrema. What if we wish to know the maximum or minimum of a function over an entire interval?

- f has a **global minimum** at p if $f(p) \leq f(x)$ for all x on a given domain.
- f has a **global maximum** at p if $f(x) \leq f(p)$ for all x on a given domain.

Ex: Where are the global minimum and maximum of the following graph?



4.3 - Optimization

To find the global (absolute) extrema of a function $f(x)$ over an interval $[a, b]$:

1. Find $a < x < b$ such that $f'(x) = 0$.
2. Evaluate f at each of these critical points.
3. Evaluate f at $x = a$ and $x = b$
4. Determine the minimum and maximum values.

Ex: Find the global extrema of $f(x) = x^3 - 12x + 1$ on $[-3, 5]$.

Ex: Find the global extrema of $f(x) = \dfrac{\ln x}{x}$ on $[1, 3]$.

Ex: Find the global extrema of $f(x) = \dfrac{\cos x}{2 + \sin x}$ on $[0, 2\pi]$.

4.3 - Optimization

Ex: Find the best possible bound for the following functions

a. $x^3 - \frac{45}{4}x^2 + 27x + 1$ for $0 \leq x \leq 8$.

b. $x + \sin x$ for $0 \leq x \leq 2\pi$.

c. $\ln(1 + x)$ for $x \geq 0$.

Ex: When an electric current passes through two resistors with resistance r_1 and r_2, connected in parallel, the combined resistance, R, can be calculated from

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

where R, r_1, and r_2 are positive. Assume that r_2 is constant.

a. Show that R is an increasing function of r_1.

b. Where on the interval $a \leq r_1 \leq b$ does R take its maximum value?

4.3 - Optimization

Ex: Two points on the curve

$$y = \frac{x^3}{1 + x^4}$$

have opposite x-values, x and $-x$. Find the points making the slope of the line joining them greatest.

Ex: For the period from 1980 to 1994, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{75}{1 + 74e^{-0.6t}}$$

where the time t is measured in years since midyear 1980, so that $0 \le t \le 14$. Find the time at which the number of VCRs was increasing most rapidly.