



Chapter 5: The Definite Integral
Section 5.1 Distance Traveled
Section 5.2 Definite Integral
Section 5.3 Fundamental Theorem
Section 5.4 Theorems about Definite
Integrals

Math 124, Section 023, Fall 2007
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5.1 - Distance from Velocity

If we are given a function that represents distance, how do we find velocity? What if we are given velocity, how can we find distance?

If velocity is constant, we have

$$\text{Distance} = \text{Velocity} \times \text{Time}.$$

For now, we consider velocity as a piecewise constant function.

Ex: Suppose a car travels at $v = 50$ mph for 2 hours. What is the total distance traveled?

Ex: A car starts moving at time $t = 0$ and goes faster and faster. Its velocity is given by the below table at 1 second intervals. Give lower and upper estimates for how far the car travels during 4 seconds.

t (sec)	0	1	2	3	4
Vel (ft/sec)	0	2	5	9	13

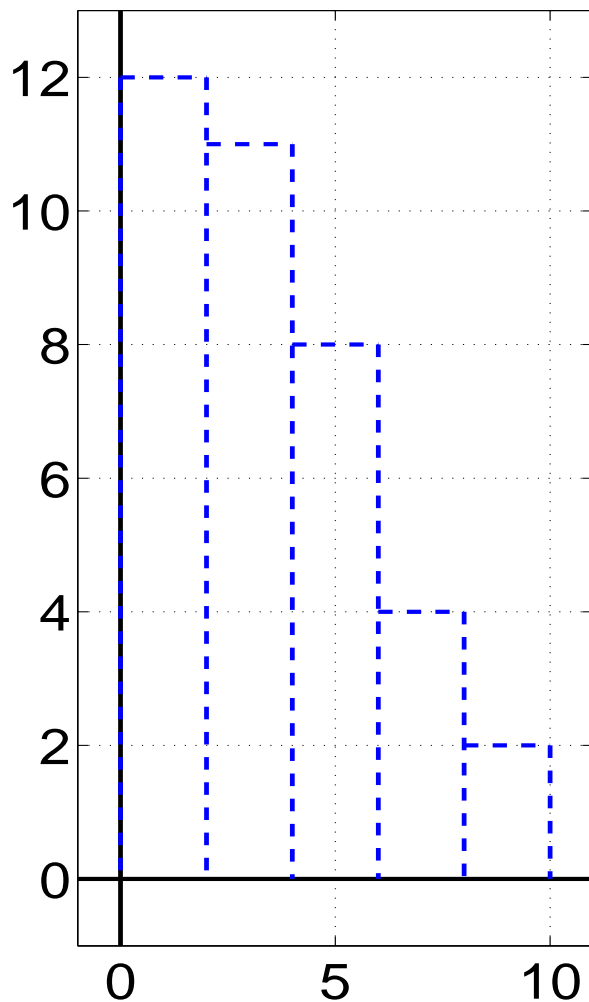


5.1 - Distance from Velocity

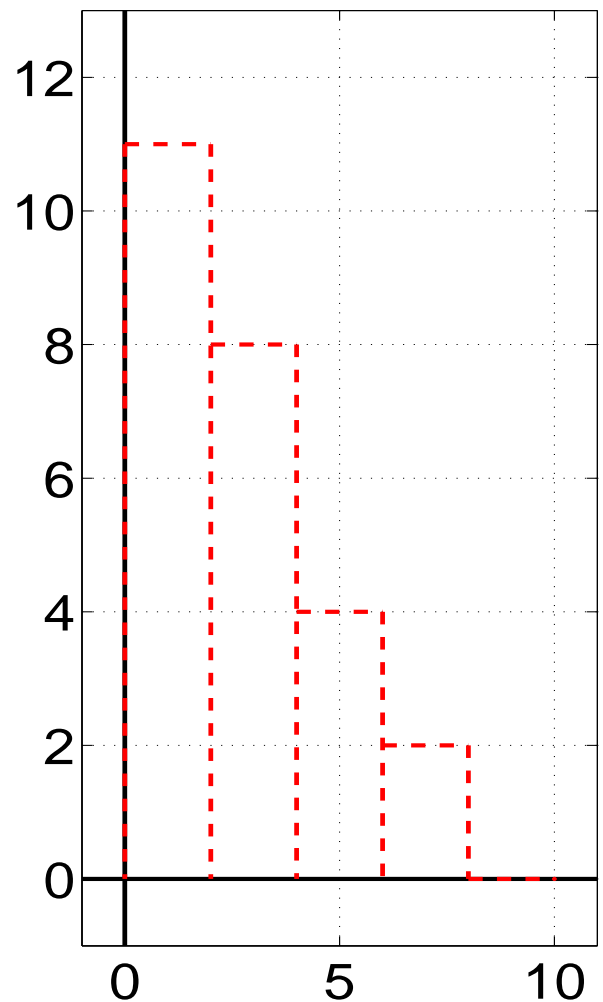
Ex: A car comes to stop twelve seconds after the driver applies the brakes. While the brakes are on, the following velocities are recorded. Find a lower and upper bound for how far the car travels during the 10 seconds.

t (sec)	0	2	4	6	8	10
Vel (ft/sec)	12	11	8	4	2	0

Upper Est



Lower Est





5.1 - Left and Right Sums

Let $v = f(t)$ be the velocity function. We take measurements of $f(t)$ at equally spaced times $a = t_0, t_1, \dots, t_n = b$. We define the interval between measurements as

$$\Delta t = \frac{b - a}{n}.$$

Left-Hand Sum: For each interval, we use the value on the left. Then the distance traveled between $t = a$ and $t = b$ is

$$\text{Dist} \approx f(t_0) \Delta t + f(t_1) \Delta t + \dots + f(t_{n-1}) \Delta t$$

Right-Hand Sum: For each interval, we use the value on the right. Then the distance traveled between $t = a$ and $t = b$ is

$$\text{Dist} \approx f(t_1) \Delta t + f(t_2) \Delta t + \dots + f(t_n) \Delta t$$

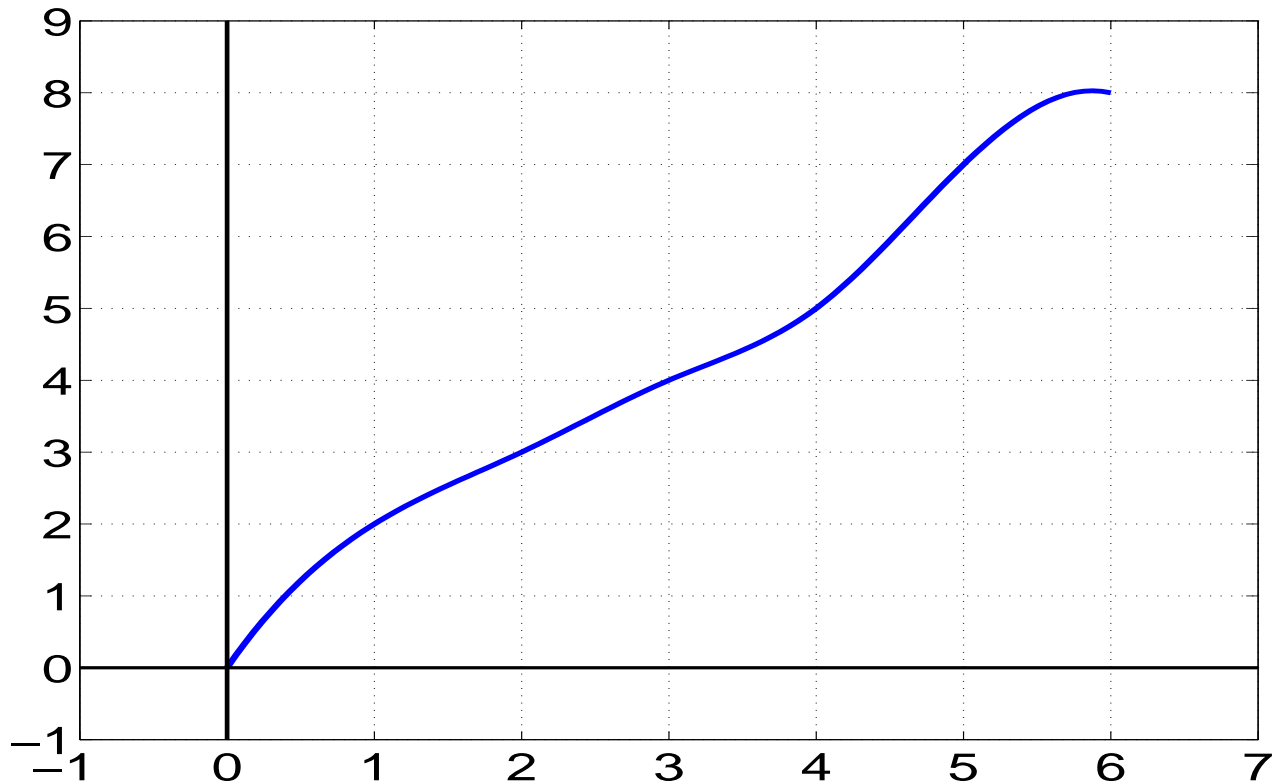
Error: The error in the sum can be approximated by the difference between the two estimates

$$\left| \begin{array}{l} \text{Difference b/t} \\ \text{upper and lower est} \end{array} \right| = |f(b) - f(a)| \cdot \Delta t.$$



5.1 - Distance from Velocity

Suppose we are given a velocity from $t = 0$ to $t = 6$ in the below graph.



For each case, estimate the distance traveled from $t = 0$ to $t = 6$. Find the size of each interval, denoted Δt .

1. Using 3 equal intervals.
2. Using 6 equal intervals.
3. Which estimate is more accurate? Why? What happens if the number of intervals approaches ∞ ?



5.1 - Distance from Velocity

Ex: The velocity of a particle moving along the x -axis is given by $f(t) = 6 - 3t$ cm/sec. Use the graph to find the exact change in position of the particle from time $t = 0$ to $t = 3$ seconds.

Ex: A baseball is thrown upwards at 96 ft/sec with velocity $v(t) = 96 - 32t$ ft/sec at time t seconds.

- a. Graph $v(t)$ from $t = 0$ to $t = 6$.
- b. When does the baseball reach the peak of its flight? How high does it go?
- c. How high is the baseball at time $t = 5$?

Ex: Yelnick McWawa rides his bike 5 miles, accelerating at a constant rate from rest to 10 mph. How long does it take for him to reach 10 mph?



5.2 - The Definite Integral

Using the same notations as Section 5.1, we may rewrite the estimates for area under the curve $f(t)$ from $a \leq t \leq b$ as

$$\text{Right-hand Sum: } \sum_{i=1}^n f(t_i) \Delta t$$

$$\text{Left-hand Sum: } \sum_{i=0}^{n-1} f(t_i) \Delta t$$

The Definite Integral

Suppose $f(t)$ is continuous for $a \leq t \leq b$.

The *definite integral* of f from a to b , written

$\int_a^b f(t) dt$, is the limit of either the left or right-hand sums with $n \rightarrow \infty$ ($\Delta t \rightarrow 0$). In other words,

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(t_i) \Delta t = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_i) \Delta t$$

Each sum is called a *Riemann sum*, f is called the *integrand* and a and b are *limits of integration*.



5.2 - Computing Definite Integrals

How would we compute $\int_1^2 x^2 dx$.

- Use an exact integral (next chapter).
- Use Riemann sums to find an approximation (need MANY terms).
- Calculate the area under the curve directly (what is area for a general curve?).
- Use numerics (calculator).

To compute $\int_a^b f(x) dx$ numerically on the calculator, we use *fnInt* (or *nInt*).

For example, to compute $\int_1^2 x^2 dx$ we enter:

$$\text{fnInt}(x^2, x, 1, 2).$$

For $\int_1^{10} \frac{1}{t} dt$ we enter:

$$\text{fnInt}(1/x, x, 1, 10).$$

Note: We can enter x no matter what. The variable in the integration is insignificant.



5.2 - Area Under the Curve

We can interpret

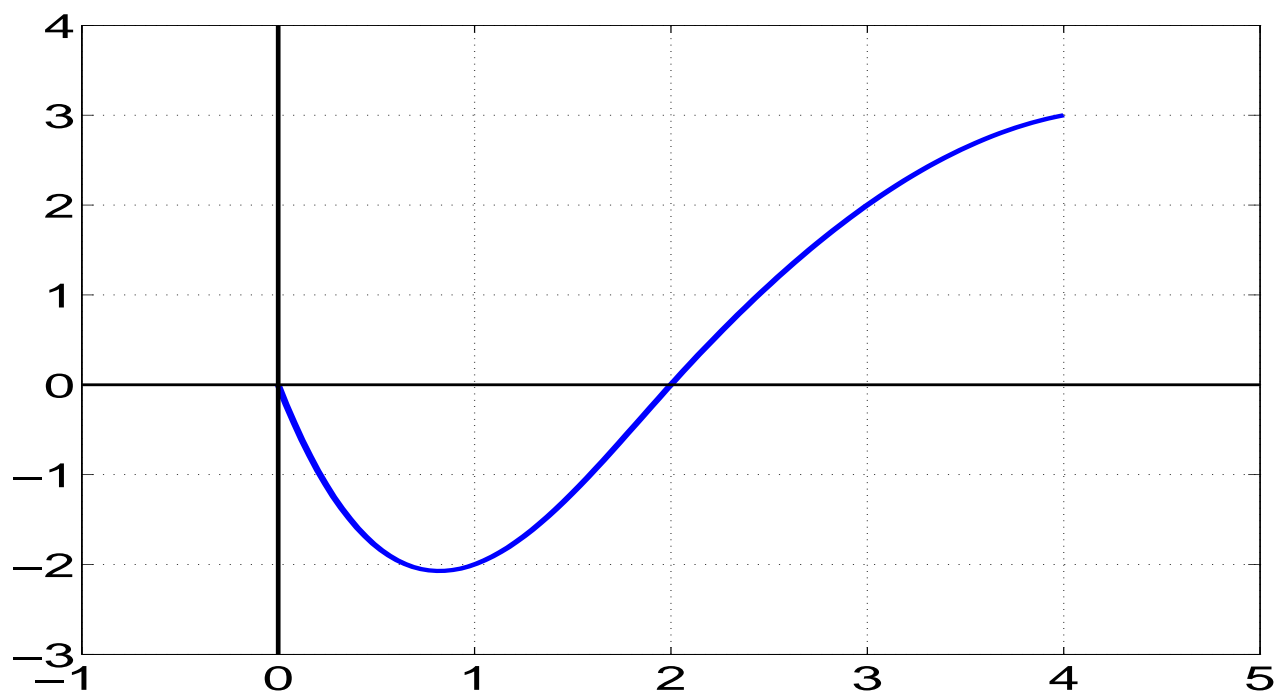
$$\int_a^b f(x) dx$$

as the area contained above the x -axis, counted positively, and the area below the x -axis, counted negatively.

Ex: Find the area under $y = \sqrt{16 - x^2}$ from $x = -4$ to $x = 4$ exactly and numerically.

Ex: Find the area under $y = \sin x$ for $x = 0$ to $x = 2\pi$ exactly.

Ex: Estimate $\int_0^4 f(x) dx$ for the below graph





5.3 - The Fundamental Thm

The Fundamental Theorem of Calculus: If f is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$F(b) - F(a) = \int_a^b f(t) dt$$

Note: This connects a function F with its derivative $F'(t) = f(t)$. The integrals in Chapter 6 depend *directly* on this results.

Ex: If you jump out of an airplane and your parachute fails to open, your downward velocity (in meters per second) t seconds after the jump is approximated by:

$$v(t) = 49(1 - (0.8187)^t)$$

Write an expression for the distance you fall in the first 5 seconds and then calculate the answer.



5.3 - The Fundamental Thm

Ex: The population of a certain city is increasing at the rate given by $P(t) = 2000e^{t+1}$ where t is time in years measured from the beginning of 1994, when the population was 10,000. Find the population at the end of 1996.

Ex: If $F(x) = \frac{1}{3}x^3$, it can be shown that $F'(x) = x^2$. Find $\int_0^1 x^2 dx$ in two ways:

- Using a calculator.
- Using the Fundamental Theorem of Calculus.



5.3 - Average Value

Let $f(t)$ be temperature at time t , in hours since midnight. What is the average temperature over 24 hours? Suppose we take n equally spaced measurements at t_1, \dots, t_n . Then for $\Delta t = 24/n$

$$\begin{aligned}\text{Avg Temp} &\approx \frac{f(t_1) + \dots + f(t_n)}{n} \\ &= \frac{1}{24} \sum_{i=1}^n f(t_i) \Delta t.\end{aligned}$$

What happens as $n \rightarrow \infty$?

$$\begin{aligned}\text{Avg Temp} &\approx \lim_{n \rightarrow \infty} \frac{f(t_1) + \dots + f(t_n)}{n} \\ &= \frac{1}{24} \int_0^{24} f(t) dt.\end{aligned}$$

Average value of f : The average value of a function f from a to b is given by

$$\frac{1}{b-a} \int_a^b f(x) dx.$$



5.3 - Average Value

Ex: What is the average value of $f(x) = \sqrt{1 - x^2}$ over the interval $0 \leq x \leq 1$?

Ex: The number of hours, H , of daylight in Madrid as a function of date is approximated by the formula

$$H = 12 + 2.4 \sin [0.0172(t - 80)] ,$$

where t is the number of days since the start of the year. Find the average number of hours of daylight in Madrid:

- a. in January
- b. over a year



5.4 - Limits of Integration

Properties of Integration Limits: We have only considered $a < b$ in the previous integrals. If f is a continuous function, then for any numbers $a, b,$ and c

$$1. \int_b^a f(x) dx = - \int_a^b f(x) dx$$

$$2. \int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx.$$

Ex: If $\int_0^2 f(x) dx = 2$ and $\int_1^2 f(x) dx = 1$ then what is $\int_0^1 f(x) dx$?

Ex: Express

$$\int_{-1}^5 f(x) dx - \int_{-1}^0 f(x) dx + \int_5^8 f(x) dx$$

as a single integral of the form $\int_a^b f(x) dx$.



5.4 - Integration of Sums

Properties of Integrand Sums: If f and g are continuous functions and c is a constant, then

- $$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$$
- $$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

Ex: Evaluate $\int_0^4 (2 - 4x) dx$ exactly.

Symmetry

If $f(x)$ is an even function and $g(x)$ is an odd function, then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

and

$$\int_{-a}^a g(x) dx = 0.$$



5.4 - Area Between Curves

If the graph of $f(x)$ lies above the graph of $g(x)$ for $a \leq x \leq b$, then the **area between f and g for $a \leq x \leq b$** is given by

$$\int_a^b (f(x) - g(x)) dx$$

Ex: Find and evaluate an integral representing the area of the region bounded by the y -axis, the line $y = 2$ and the line $y = x$.

Ex: Find the area of the region enclosed by $f(x) = x^2$ and $g(x) = 2 - x^2$.

Comparison of Definite Integrals: Let f and g be continuous functions.

1. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)..$$

2. If $f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$