Chapter 6: Constructing Antiderivatives

Section 6.1 Antiderivatives Graphically and Numerically

Section 6.2 Constructing Antiderivatives Analytically

Section 6.3 Differential Equations

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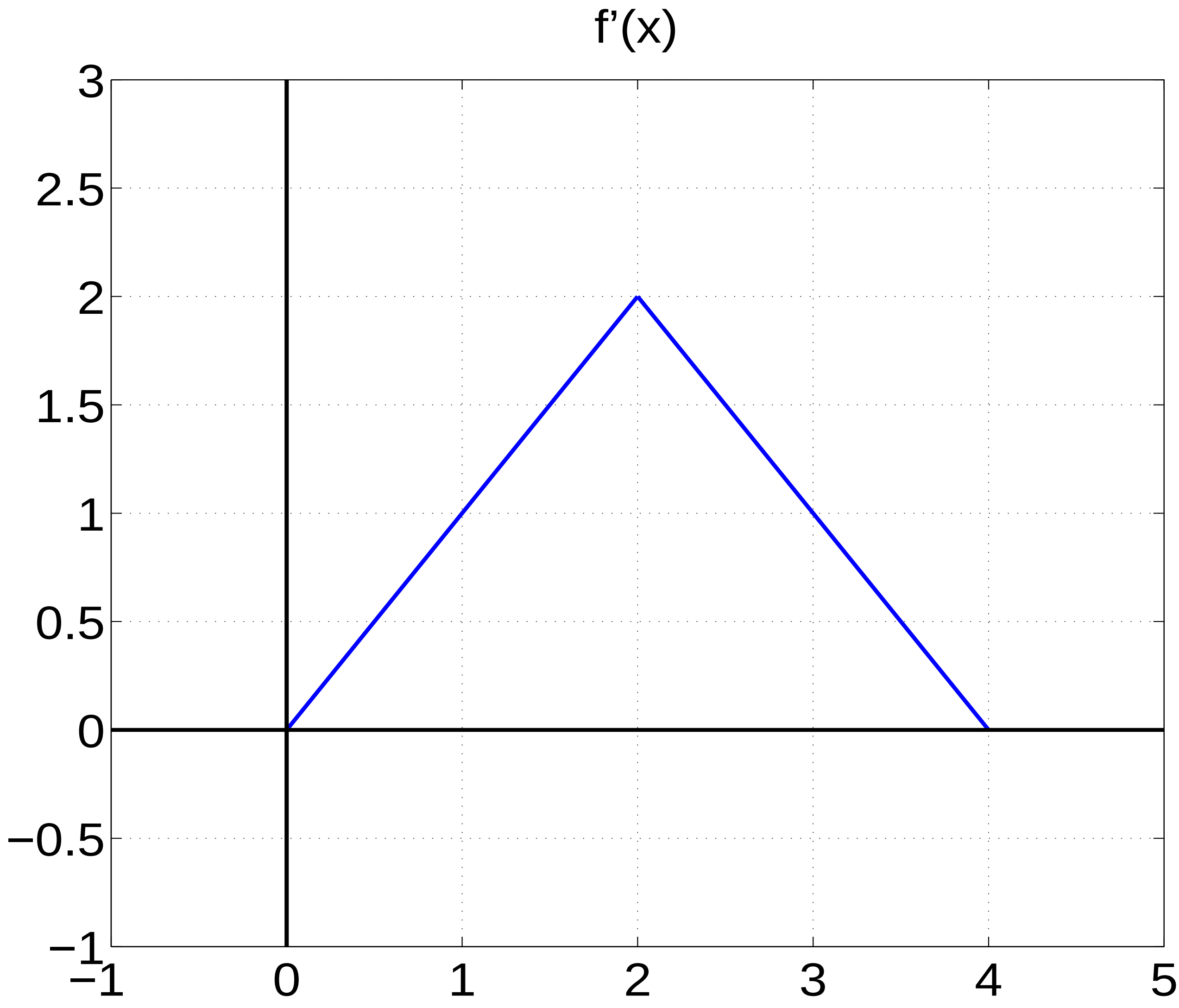
6.1 - Antiderivatives Graphically

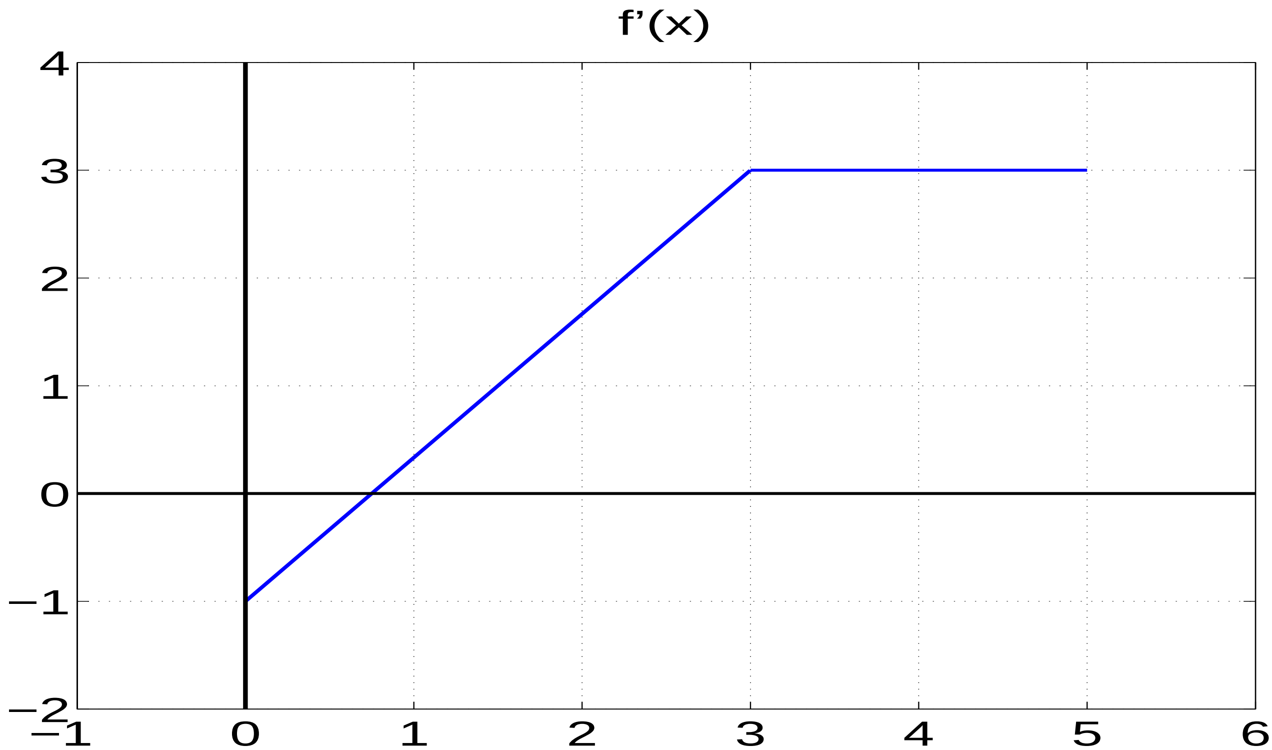
If the derivative of F is f, we call F an **antiderivative** of f. If $f = 1$ then $F = x$ is an antiderivative of $f = 1$. What about $F = x + 1$? $F = x + 2$?

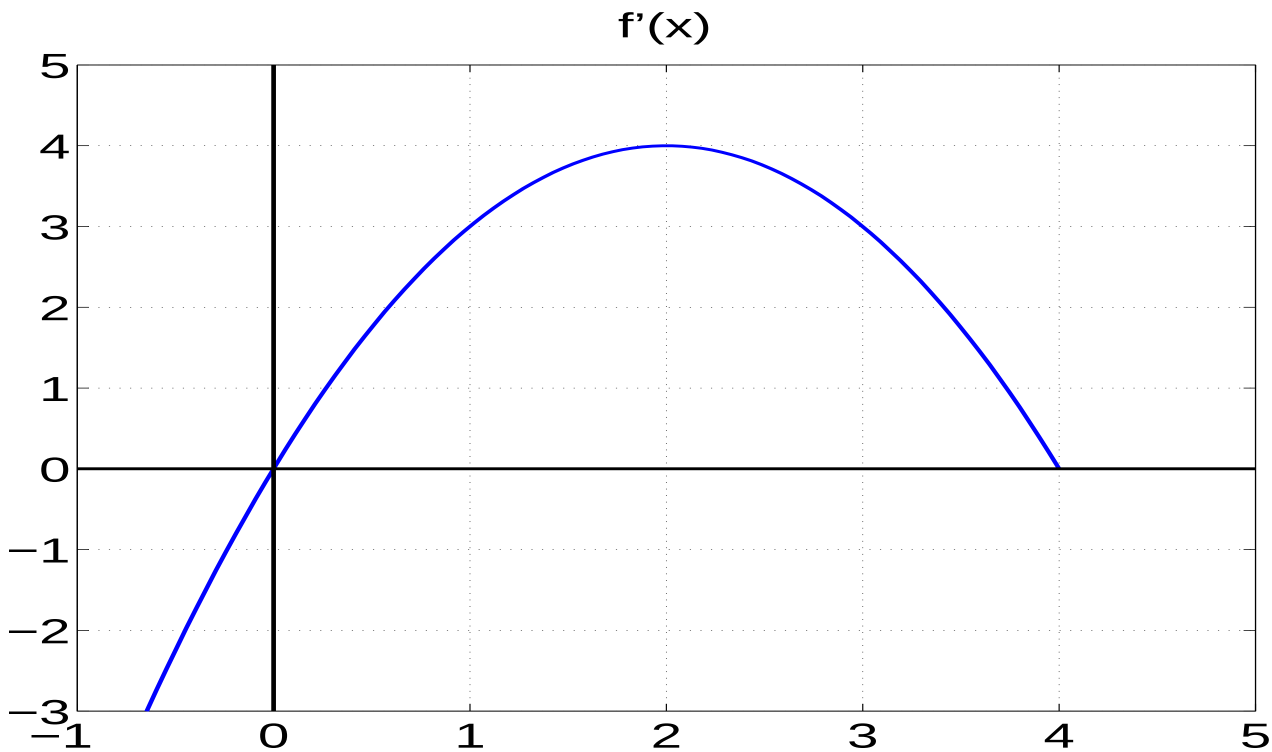
Ex: The graph of f' is given in the below figure. Sketch a graph of f in the cases when $f(0) = 0$ and $f(0) = 1$.



6.1 - Antiderivatives Graphically

Ex:For each of the following, f' is given by the figure. Sketch a graph of f in the cases when $f(0) = 0$ and $f(0) = 1$. Indicate local min/max and inflection points of f.



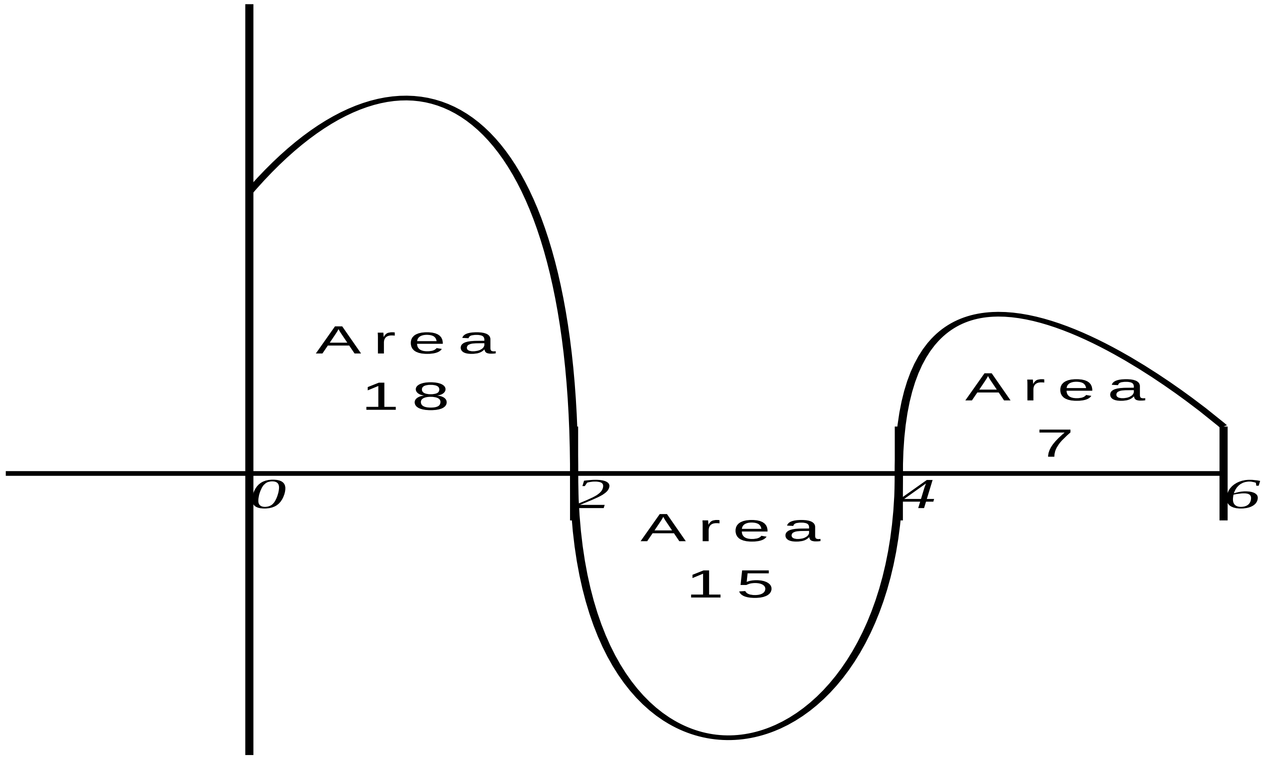


6.1 - Antiderivatives Numerically

We use the Fundamental Theorem of Calculus to determine integrals. Recall

$$\int_a^b f'(x) = f(b) - f(a).$$

Ex: Given the graph of f' below and that $f(0) = -5$, find $f(2)$, $f(4)$ and $f(6)$.



6.2 - Indefinite Integrals

In this section, we present the general antiderivative for numerous functions. For an antiderivative, we often say $F(x)$ *is the antiderivative of* $f(x) = F'(x)$. Instead of using this notation, we define the **indefinite integral**

$$\int f(x)\,dx = F(x) + C.$$

Note: This is the integral of $f(x)$, but with NO limits. For this section, we denote C as a general constant.

Antiderivative of $\mathbf{f(x) = 0}$

Since $\dfrac{d}{dx}C = 0$ we have

$$\int 0\,dx = C.$$

Antiderivative of $\mathbf{f(x) = k}$

For any constant k, since $\dfrac{d}{dx}kx = k$ we have

$$\int k\,dx = kx + C.$$

6.2 - Indefinite Integrals

Antiderivative of $\mathbf{f(x) = x^n}$

For $n \neq -1$, since $\frac{d}{dx}x^{n+1} + C = (n+1)x^n$ we have

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Antiderivative of $\mathbf{f(x) = 1/x}$

Since $\frac{d}{dx}\ln(x) = \frac{1}{x}$ we have

$$\int \frac{1}{x} \, dx = \ln|x| + C.$$

Note: We have $\ln|x|$ not $\ln(x)$ since $\frac{1}{x}$ is well defined for $x < 0$ as well; thus, its integral has to be well defined.

6.2 - Indefinite Integrals

Antiderivative of $\mathbf{f(x) = e^x}$

Since $\dfrac{d}{dx}e^x = e^x$ we have

$$\int e^x\,dx = e^x + C.$$

Antiderivative of $\mathbf{f(x) = \sin x}$ and $\mathbf{f(x) = \cos x}$

Since $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\cos x = -\sin x$ we have

$$\int \sin x\,dx = -\cos x + C$$

and

$$\int \cos x\,dx = \sin x + C.$$

6.2 - Indefinite Integrals

Just as with derivatives, sums and constant multiples can be separated in integrals

$$\int (f(x) \pm g(x))\ dx = \int f(x)\,dx \pm \int g(x)\,dx$$

and

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

Ex: Evaluate the following integrals

a. $\int \left(x^4 + 5\right)\ dx$.

b. $\int \left(\sqrt{y} + \frac{1}{y^2} + \frac{1}{y}\right)\ dy$.

c. $\int e^r\ dr$.

d. $\int (\cos(q) + \sin(t))\ dq$.

6.2 - Application of the FToC

Now that we can evaluate indefinite integrals, we can use the Fundamental Theorem of Calculus to evaluate definite integrals exactly. Recall

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Ex: Use the FToC to evaluate the following integrals

a. $\displaystyle\int_0^4 6x\,dx.$

b. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx.$

c. $\displaystyle\int_0^2 \left(\frac{x^3}{3} + 2x\right)\,dx.$

d. $\displaystyle\int_0^1 2e^z\,dz.$

6.3 - Differential Equations

A **differential equation** is an equation involving the derivative of an unknown function. Some examples:

$$\frac{dy}{dt} = 4, \quad x\frac{dy}{dx} = 3y, \quad y' = 3xy.$$

A differential equation of the form

$$\frac{dy}{dx} = f'(x)$$

has a general solution of $y(x) = f(x) + C$.

When we are given some conditions on the differential equation to determine the specific solution, then we call it an *initial value problem*. For example, if

$$\frac{dy}{dx} = 2x, \;\; y(0) = 1$$

then we have a general solution of $y(x) = x^2 + C$. Since $y(0) = 1$ we have that $C = 1$.

6.3 - Differential Equations

Ex: Determine if $y = e^{x^2}$ is a solution to the differential equation $y' = 2xy$.

Ex: Verify that $P = 50 + Ce^{0.20t}$ is a general solution to

$$\frac{dP}{dt} = 0.20P - 10.$$

Find the particular solution satisfying the initial condition $P = 60$ when $t = 0$.

Ex: Is there a value of n which makes $y = x^n$ a solution to the equation

$$13x\frac{dy}{dx} = y?$$

Ex: What relationship does the following imply

$$\frac{dy}{dx} = ky?$$

What is the general solution to this equation?

6.3 - Differential Equations

Ex: A tomato is thrown upward from a bridge 25 m above the ground at 40 m/sec.

a. Give formulas for the acceleration, velocity, and height of the tomato at time t.

b. How high does the tomato go, and when does it reach its highest point?

c. How long is it in the air?

Ex: A car starts from rest at time $t = 0$ and accelerates at $-0.6t + 4$ meters/sec^2 for $0 \leq t \leq 12$. How long does it take for the car to go 100 meters?