Chapter 6: Constructing Antiderivatives

Section 6.4 Second Fundamental Theorem of Calculus

Chapter 7: Integration

Section 7.1 Integration by Substitution

Math 124, Section 023, Fall 2007

Instructor: Paul Dostert

6.4 - 2^{nd} FToC

The second fundamental theorem of calculus is essentially a restatement of the original FToC. The idea is that if we want a function of x that is the antiderivative of f, then we can simply integrate f from some number a to x.

Second Fundamental Theorem of Calculus: If f is a continuous function on an interval, and if a is any number in that interval, then the function F defined as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)\,dt.$$

Note: This is the first FToC, but we allow F to be a function of x, rather than just a number. Think of this as an integral formula where an endpoint (the x endpoint) can change.

6.4 - 2^{nd} FToC

The important use of the 2^{nd} FToC is in taking derivatives of integrals. Clearly

$$\frac{d}{dx}\int_a^b f(t)\,dt = 0$$

because the integral is simply a constant. What is

$$\frac{d}{dx}\int_a^x f(t)\,dt?$$

Well, since we are integrating $f(t)$, we have

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

where F is an antiderivative of f. Thus

$$\frac{d}{dx}\int_a^x f(t)\,dt = F'(x) - F'(a) = f(x).$$

Simply put, the derivative of the integral is the function itself.

6.4 - 2nd FToC

Note: The previous statement applies only if the limit in the integral is the variable we are taking the derivative of. For example

$$\frac{d}{dx}\int_a^{x^2} f(t)\,dt \neq f(x).$$

What does it equal? Well, we use the chain rule

$$\begin{aligned}\frac{d}{dx}\int_a^{x^2} f(t)\,dt &= \frac{d}{dx}\left(F(x^2) - F(a)\right)\\ &= 2x \cdot f(x^2).\end{aligned}$$

Ex: Evaluate the following

a. $\frac{d}{dx}\int_0^{x} \sin(t)\,dt.$

b. $\frac{d}{dz}\int_5^{z} e^{2t\cos(t)}\,dt.$

c. $\frac{d}{dx}\int_x^{2} y^2 + y - 1\,dy.$

6.4 - 2^{nd} FToC

Ex: Write an expression for f with the given properties

a. $f'(x) = \cos\left(x^3\right),\ f(0) = 1$.

b. $f'(x) = \ln\left(x^2+1\right),\ f(5) = 18$.

c. $f'(z) = e^{z+\arctan z^2},\ f(1) = 4$.

Ex: Evaluate the following

a. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^2 + t\,dt.$

b. $\dfrac{d}{dy}\displaystyle\int_{\sin y}^{\cos y} e^{t+1}\,dt.$

c. $\dfrac{d}{dt}\displaystyle\int_{t^6}^{3} \ln\left(x^2\right)\,dx$.

7.1 - Integration by Substitution

Recall, from the chain rule, we have

$$\frac{d}{dx}\sin\left(x^2\right) = 2x\cos\left(x^2\right).$$

Then what is

$$\int x\cos\left(x^2\right)\,dx?$$

We can think of x^2 as an *inside* function and x as an *outside* function. The derivative of the inside function is found in the integral.

To find

$$\int x\cos\left(x^2\right)\,dx.$$

we can choose $u = x^2$ to be the inside function. Then $du = 2x \cdot dx$. To rewrite the integral in terms of u, we need a $2x \cdot dx$ term. We write

$$\int \frac{1}{2}\cos\left(x^2\right)2x\,dx.$$

We now rewrite the integral in terms of u

$$\frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin\left(x^2\right) + C.$$

7.1 - Integration by Substitution

We can think of integration by substitution as a backwards chain rule, where we are searching for the inner function from chain rule.

Ex: Evaluate the following

a. $\int x\left(x^2-1\right)^3\,dx$.

b. $\int \sec^2 2\theta\,d\theta$.

c. $\int \cos t\,\cos(\sin t)\,dt$.

d. $\int e^x \sin(e^x)\,dx$.

e. $\int \frac{x}{1+x^4}\,dx$.

7.1 - Integration by Substitution

Sometimes it is simply not possible to find the integral via substitution. For example,

$$\int x^3 \sin\left(x^2+1\right)\, dx.$$

What would our options for substitution be? Why don't these work? What do we do?

Sometimes we may have a function that is it not obvious will work. What do we do then? We guess! For example,

$$\int x\sqrt{x+1}\, dx.$$

Clearly this doesn't come directly from a chain rule derivative. We still treat the problem the same. We choose $u = x + 1$, thus $du = dx$. We have

$$\begin{aligned}\int \sqrt{u}(u-1)du &= \int \left(u^{3/2} - u^{1/2}\right)\, du \\ &= \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C \\ &= \tfrac{2}{5}(x+1)^{5/2} - \tfrac{2}{3}(x+1)^{3/2} + C\end{aligned}$$

7.1 - Integration by Substitution

Ex: Evaluate the following

a. $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx.$

b. $\displaystyle\int \left(x^3+1\right)^{1/3} x^5\,dx.$

Ex: Evaluate the following definite integrals

a. $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx.$

b. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}\,dx.$

c. $\displaystyle\int_0^1 t^2 2^{-t^3}\,dt$

d. $\displaystyle\int_0^{1/2} \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx.$