review of James Robert Brown, Philosophy of Mathematics, An Introduction to the World of Proofs and Pictures, Routledge, New York, 1999.

reviewed by William G. Faris, Department of Mathematics, University of Arizona, faris@math.arizona.edu

Brown's book is mainly about the philosophy of mathematics, but a special theme is pictures and mathematical proof. Here is an example. Two mathematicians each wish to convey an understanding of the intermediate value theorem, which states that a continuous function on a closed bounded interval that starts below the horizontal axis and arrives above the axis must cross the axis at some point. The first mathematician draws a picture and claims that this gives a proof. The picture is not itself the proof, but it acts as a symbol that extracts the essence of the general situation. The second mathematician gives a proof in the usual language of analysis, following the argument of Bolzano. This begins by assuming that f is continuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$. The set S consisting of all x with $a \leq x \leq b$ and $f(x) < 0$ is non-empty and bounded above. So it has a least upper bound c. And the proof goes on to conclude eventually that $f(c) = 0$.

The second mathematician claims that the Bolzano proof justifies the picture proof. The first mathematician retorts that it is the other way around. The picture proof captures what we want a continuous function to do. It follows only then that the particular mode of formalization used in the Bolzano proof is correct, since it reconstructs this behavior. There is a Platonic world of absolute mathematical truth, and the picture proof is a successful way of capturing it, as good as or better than any other.

This example indicates that there is more than one way to look at the role of pictures in proof. The weakest claim is that a picture can serve as an aid to inspire a proof. The mathematician presenting the Bolzano proof could agree to this. A stronger position is that a picture may be taken to be an element of a formal proof. If one can specify syntactic rules for words that constitute a proof, then one might also be able to specify syntactic rules for pictures that constitute a proof. There are several recent works with this theme [1][2]. The strongest position is that of the mathematician presenting the sketch of the function crossing the axis. This sketch is supposed to capture a typical representation of the true situation. It is a way to perceive the Platonic world of mathematical reality.

According to Brown, a proponent of this point of view, "Platonism asserts the existence of abstract entities; it says that numbers, functions, rules and so on are just as real as trees and electrons, though they are not physical entities located in space and time." He celebrates this with a quip. "Being a *mere* mathematical entity is not some second-rate status. I would take great pride in being an integer—if that made any sense."

Brown's book is partly expository, but it also takes a position. Its stated goals are:

a. to introduce readers to the philosophy of mathematics;

b. to introduce some of the newer issues, those associated with mathematical experiments, the use of computers, and especially visualization;

c. to argue for Platonism in the philosophy of mathematics (and in particular for a Platonistic view of how pictures work).

The book is not aimed at the general reader; words such as "epistemic" and "ontic" occur without explanation. However the prose is readable, and the author gives a number of examples of picture proofs. Mathematicians will find it a cheerful introduction to the traditional positions on the foundations of their subject. Unfortunately, the book is unsystematic and at times imprecise. These shortcomings are particularly striking in the treatment of picture proofs. For example, on page 4 there is a diagram that is supposed to illustrate the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ with infinite sum. The author says that "the picture does not give us an inkling of this startling result." In fact, the picture does not illustrate the harmonic series but instead illustrates the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ with sum 2. If the author had drawn the correct picture, then the analytic argument on page 5 that the sum of the harmonic series is infinite could easily be demonstrated graphically.

Later, on pages 36 and 37, the author refers to the fact that the sum $s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ has the value 1. A picture proof is contrasted favorably with a cumbersome "traditional" proof. However the proof based on the solution of $2s = 1 + s$ is at least as succinct and rigorous as the picture proof. (Of course one has to rule out the solution $s = \infty$.)

This review will concentrate more on the philosophy of mathematics than on the issues associated with the use of pictures. Here the author's arguments may have an unintended consequence. While mathematicians are often natural Platonists, exposure to Platonist arguments may bring new doubts. A defense of an implicitly held position can weaken the position, precisely by making it more explicit. In this case this may result from the lack of a detailed theory of the Platonic world and of how we come to perceive it.

There are various views on the philosophy of mathematics, and it is striking that no clean resolution has been achieved. A person's view on this subject should presumably come from reasoning and experience. It may also depend on the domain where the person feels most comfortable. Here are four possibilities for such a domain.

1. The Ideal

In Plato's philosophy the ideal world is the world of patterns or archetypes of which the world of experienced phenomena is only an imperfect replica. Contemporary *Platonism* in the philosophy of mathematics might not make such strong claims about the ideal, but it would at least assert that mathematical objects have genuine existence independent of us—mathematical entities are neither ideas nor material things, but belong to yet another realm. Thus there are infinitely many natural numbers, and their nature is accessible to the human mind. A number such as 2 is as real as a house with that address, but somehow more permanent. *Structuralism* is a newer but related view of the nature of mathematics. It asserts that mathematical entities do not have existence in their own right, but only as places in structures. This is intended to answer the puzzle of why we seem to be unable to say exactly what object the number 2 is. Suppose 0 is the empty set. Then the Zermelo natural numbers are 0, $1 = \{0\}$, $2 = \{1\}$, and so on, with $n = \{n - 1\}$ for each $n \neq 0$. On the other hand, the von Neumann natural numbers are 0, $1 = \{0\}$, $2 = \{0, 1\}$, and so on, with $n = \{0, 1, \ldots, n - 1\}$ in the general case. The Zermelo 2 is not the von Neumann 2. However the Zermelo natural numbers and the von Neumann natural numbers are realizations of one and the same natural-number structure (that is, in more familiar terms, they are isomorphic). The number 2 is neither $\{1\}$ nor $\{0, 1\}$, but the place (third from the left) each of them occupies in their respective realization of the natural-number structure. Of course, that leaves open the question of what structures are. The shift from Platonism to structuralism may be analogous to changing the mathematical perspective from set theory to category theory. The intuition behind set theory is that of a collection of objects, while category theory gives more emphasis to the mathematical operations, considered abstractly. It is no longer important to think of natural numbers as objects. The categorical perspective might instead emphasize the successor operation, so the next number indicates the next house. Whether this corresponds to a significant philosophical shift is less clear. Is the successor function not also a mathematical object? This question about the nature of mathematical entities has a psychological counterpart. People may first see functions only as actions on other mathematical objects, then later come to see them instead as objects in their own right.

2. Nature

A realistic theory of mathematics need not be a priori, it can also be empirical. *Naturalism* is the view that mathematical statements are about regularities in the physical world. This poses the question of how to understand the seeming permanence of mathematics. Are facts about the number 16, for instance, merely the result of experiment? Could they change? Not necessarily. Regularities in the physical world are, presumably, independent of us, though they are—to the extent they are known to us at all—known in large part as the result of experiment. Thus, on a naturalist view, facts about the number 16 are not a result of experiment, any more than Newton's law of gravitation, had it turned out to be a law, would have been the result of experiment. Physical laws are usually thought not to change, and so there is no reason to think that mathematical laws might be changeable. Another puzzle for naturalism is how mathematics can arrive at properties of infinite sets. According to one view, extrapolations of properties of large finite sets can be the basis of our understanding of infinite sets [3]. This might even include the theory of infinite cardinal numbers, since insight into this subject depends on properties of the exponential function that hold both in the finite or infinite case. The main advantage of naturalism is that it could give an indication of why mathematics is so successful in describing nature. If mathematics is part of nature, then it should certainly reflect nature.

3. Construction

There are also the variants of constructivism. These are points of view that limit the sphere of mathematics in some way. The philosophical motivation for imposing such limitations may vary. However in all cases these theories seek to reformulate mathematics in terms that are more concrete and computational. *Intuitionism* is the view that mathematical objects are mental constructions, not independent of us. It maintains that each natural number is humanly conceivable, but rejects the idea that the infinite totality of all natural numbers is humanly conceivable. There is no problem with 16, nor with $2^{16} = 65536$, nor for that matter with $2^{2^{16}}$. Other versions of constructivism may be more limiting. One extreme is *radical finitism.* A skeptic can doubt not only the existence of the infinite set of natural numbers, but also the existence of large finite numbers. One can arrange 65536 numbers (or houses) in a row, but there is no way doing the same with $2^{2^{16}}$, counting by ones [4]. Maybe this view will come to seem congenial in the computer age. Brown shows some sympathy for it. In the course of describing one of the less radical forms of constructivism, he declaims: "Give us the power of Platonism or give us computational practicality. Constructivists give us neither."

4. The Printed Page

Finally, there are nominalist and conventionalist views. Mathematical symbols have no referents; mathematics is manipulation of symbols according to rules. One variant of this is *formalism.* In Hilbert's version of formalism the consistency of the symbolic manipulation underlying arithmetic was supposed to be justified by finitist reasoning. The second Gödel theorem put an end to this hope. However the theorem does not refute formalism; it only makes it less comfortable.

This reviewer has so far not succeeded in perceiving Platonic reality. Even his intuition of the finite seems weak. So his hope is that some blend of the naturalistic and formalistic theories will help him understand the workings of mathematics. Brown gives space to other views, but all his sympathy is with the Platonist position. So the remainder of this review will mainly deal with this view.

Platonism is stated in classic form by Frege, who writes (quoted by Brown) that mathematical entities are "neither things of the outer world nor of ideas." He continues:

"A third realm must be recognized. What belongs to this corresponds to ideas, in that it cannot be perceived by the senses, but with things, in that it needs no bearer to the contents of whose consciousness to belong. Thus the thought, for example, which we express in the Pythagorean theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no bearer. It is not true for the first time when it is discovered, but is like a planet which, already before anyone has seen it, has been in interaction with other planets."

Frege's words describe the ontic dimensions of Platonism. Mathematical entities are real, and they exist independently of us and outside of space and time. Brown's discussion also attempts to clarify the epistemic dimensions of Platonism. His claim is that we have access to the mathematical realm that is something like our perceptual access to the physical realm.

At first this position seems natural and obvious. Mathematics students are taught that there is a set of natural numbers. It is certainly there independent of their wishes. The set is infinite, which argues against it being part of the natural world. And one can—perhaps—have a clear conception of it. On the other hand, this certainty can be shaken by further reflection. What is a natural number, exactly? Students in set theory courses are most often taught the von Neumann natural numbers. As mentioned above, in this account the number n is a particular set with n elements, namely $\{0, 1, \ldots, n-1\}$. This is clever and has been influential; a device much like this occurs in the C programming language. However, does anyone believe that this is what numbers are? Certainly every set has a cardinal number, but any particular set construction of this number seems arbitrary. There is the Russell definition, in which 2 is the class of all sets with two elements, but this seems extravagant. For that matter, what is a set? What kinds of objects can belong to a set? Is it really possible to collect infinitely many objects into a set? If so, how can one experience an infinite set? I may not perceive each element of the set, but I can perhaps get a clear idea of its properties. However there is also a problem with this. Mathematicians often think that they have an established theory of natural numbers. However Abraham Robinson (using formalist methods) constructed a new theory of natural numbers that captures much of the same arithmetic, but that distinguishes between standard and non-standard natural numbers. Each natural number $0, 1, 2, 3, \ldots$ with a name is a standard natural number. There are non-standard natural numbers, and each non-standard natural number is greater than each standard natural number. At first one might think that the natural numbers of the Platonic realm are the standard natural numbers. But there is a more radical view: the natural numbers of the Platonic realm are those of the Robinson theory, and Robinson's achievement was to perceive a new distinction in this realm. (The use of this new predicate "standard" has been compared to color on a TV set: the picture is the same, but we see distinctions that we could not make before [5].) It is not clear what kind of Platonic perception would allow us to see which view is true. There is even the possibility that the Platonic realm contains several flavors of natural numbers. How can one investigate such matters?

This question does not have a good answer in the Platonist framework, but here the formalist position is stronger. For instance, the following passage contrasts the conventional theory with the Robinson theory from a formalist point of view [5]. The context is a sequence of observations, to be treated by the methods of probability theory. In the first theory the sequence is infinite; in the second it is finite, up to some non-standard natural number.

"The conventional approach involves an idealization, because one cannot actually complete an infinite number of observations. The second approach also involves an idealization, because one cannot actually complete a non-standard number of observations. In fact, it is in the nature of mathematics to deal with idealizations. The choice of a formalism must be based on esthetic considerations, such as directness of expression, simplicity, and power. Actually, different formalisms in no way exclude each other, and it can be illuminating to look at familiar material from a fresh point of view."

Brown makes the strongest possible claims for Platonism at the beginning of the book (pp. 11–14). Later on, in a few brief paragraphs (p. 150), he seems to retreat. At this point he distinguishes external realism (or metaphysical realism) from internal realism (or scientific realism). External realism is "the doctrine that statements about numbers, rules, trees, and electrons are true (or false) independently of our beliefs, our evidence, our conceptual structure, our biology, our ways of testing." Internal realism says that "the theoretical entities of science (electrons, genes) have the same ontological status as observable entities (trees and cloud chambers)." The connection with Platonism is the following. "Numbers, functions and rules are just as real as trees and electrons, though not metaphysically real. To say that they are real is just to say that under ideal conditions theories involving abstract entities are confirmed in the same way any other theory is confirmed. Thus Platonism is like scientific realism. Both can be taken in the internal realist way and be sharply distinguished from metaphysical realism."

It may not be much of a compliment to a natural number to say that it is just as real as an electron. After all, in quantum mechanics an electron does not even have a position, until it is observed. (Could one take great pride in being an electron?) It is thus amusing that a crucial step in Brown's defense of Platonism depends on quantum mechanics. This step arises in dealing with the following problem. According to one plausible theory of knowledge, there must be a causal connection between the object known and the knower. However, since abstract objects exist outside space and time, they cannot interact causally with us, and so we can never know them.

To save Platonism, Brown must first refute this causal theory of knowledge. This refutation can be accomplished if he can give even a single example where knowledge is obtained in a non-causal way. His argument involves quantum mechanics. Two particles in a state with total spin zero are carefully separated and moved to distant locations. For each particle and each axis there is a way of setting up an experiment to measure the spin component of the particle along the axis. This component can have only two values, a plus value and a minus value. Each of these values for one particle has the same probability, that is, one half. However the values for the two particles are correlated. In fact, when the components for the two particles are along the same axis, they always have opposite signs. When different axes are used, the components are still highly correlated, though no longer perfectly.

A first physicist is to measure a spin component of the first particle. A second physicist at the distant location is to measure a spin component of the second particle. Since the measurements are simultaneous and the locations are distant, there is no direct causal connection that links the results. However, according to the famous analysis of Bell, the correlations are so strong that they also have no common cause in the past. (The details of this analysis may be found, for instance, in the appendix to David Wick's book [6]). In the case when the two physicists use the same axis, the second physicist can measure the second particle and gain instantaneous knowledge of what the first physicist found with the first particle. This knowledge is gained reliably, but at least one link in the process proceeds without any kind of causal connection.

Of course, in some other theory of knowledge it may be enough to obtain knowledge by a reliable mechanism, even if it is not causal. This poses the question of whether it is possible to devise a reliable mechanism for acquiring knowledge of the Platonic realm. In any case, what Brown argues is that the causal theory of knowledge presents no obstacle to perception of the Platonic universe. One would still like a detailed account of how this perception takes place. But never mind. The vision is there: Platonic paradise. The natural numbers (infinitely many of them) exist outside of space and time, we know them and their properties, but this knowledge is not caused. The spirit of this is captured in a literary image [6]:

"The belief is handed down in Beersheba: that, suspended in the heavens, is another Beersheba, where the city's most elevated virtues and sentiments are poised, and that if the terrestrial Beersheba will take the celestial one as its model the two cities will become one. The image propagated by tradition is that of a city of pure gold, with silver locks and diamond gates, a jewel-city, all inset and inlaid, as a maximum of laborious study might produce when applied to materials of the maximum worth. True to this belief, Beersheba's inhabitants honor everything that suggests for them the celestial city: they accumulate noble metals and rare stones, they renounce all ephemeral excesses, they develop forms of composite composure."

Acknowledgment: The reviewer thanks Olga Yiparaki and Shaughan Lavine for perceptive comments.

References

[1] Gerard Allwein and Jon Barwise (editors), Logical Reasoning with Diagrams, Oxford University Press, New York, 1996.

[2] Sun-Joo Shin, The Logical Status of Diagrams, Cambridge University Press, New York, 1994.

[3] Shaughan Lavine, Understanding the Infinite, Harvard University Press, Cambridge, MA, 1994.

[4] Edward Nelson, Predicative Arithmetic, Princeton University Press, Princeton, NJ, 1986.

[5] Edward Nelson, Radically Elementary Probability Theory, Princeton University Press, Princeton, NJ, 1987.

[6] David Wick, The Infamous Boundary: Seven Decades of Controversy in Quantum Physics, Springer (Copernicus), 1996.

[7] Italo Calvino, Invisible Cities, Harcourt, San Diego, 1978.