

Generating Functions

Notes for Math 447

March 31, 2011

1 Ordinary generating functions

1.1 One variable

1.1.1 Generalities

An ordinary generating function is a convenient way of working with a sequence of numbers a_k defined for $k \geq 0$. It is a power series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k. \quad (1)$$

The utility of this notion is explained by the product property. Consider a collection of functions $f^{(j)}(x)$ for j in some index set B . For notational simplicity we take B to be the natural numbers from 1 to n . Suppose also that

$$f^{(j)}(x) = \sum_{k=0}^{\infty} a_k^{(j)} x^k. \quad (2)$$

Let

$$f(x) = f^{(1)}(x)f^{(2)}(x)\cdots f^{(n)}(x) = \sum_{k=0}^{\infty} a_k x^k. \quad (3)$$

be the product of the ordinary generating functions. Then the coefficient a_k corresponding to the product is

$$a_k = \sum_{k_1+k_2+\dots+k_n=k} a_{k_1}^{(1)} a_{k_2}^{(2)} \cdots a_{k_n}^{(n)}. \quad (4)$$

This is a sum over all sequences k_1, \dots, k_n of natural numbers with $k_1 + k_2 + \dots + k_n = k$ of combinatorial coefficients satisfying a product rule. In fact, it is a sum over multi-indices. So a complicated operation involving a huge sum over multi-indices corresponds to a simple operation on the ordinary generating function: multiplication.

One use for ordinary generating functions is to count multi-indices on a set B satisfying certain restrictions. Say that one wants to know how many ways there

are to select up to 2 apples, 3 or 4 oranges, and an even number of bananas to get k fruits. Here B is the set of three fruits. The ordinary generating function is

$$f(x) = (1+x+x^2)(x^3+x^4)(1+x^2+x^4+x^6+x^8+\dots) = \frac{1-x^3}{1-x}x^3(1+x)\frac{1}{1-x^2}. \quad (5)$$

The factors correspond to the different kinds of fruits; the terms within the factors give the inventory of available numbers of each kind of fruit.

1.1.2 Multiset coefficients

An important special case is when there are no restrictions on the multi-indices. In that case, the ordinary generating function for the number of multi-indices on an n point set of size k is

$$M_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x+x^2+x^3+\dots)^n = (1-x)^{-n}. \quad (6)$$

1.1.3 Binomial coefficients

Another important special case is when the restriction is that the values of the multi-index must be 0 or 1. Such multi-indices correspond to subsets of B . In that case, the ordinary generating function for the number of subsets of an n point set of size k is

$$B_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n. \quad (7)$$

Sometimes a problem that does not seem to be a multi-index problem can be reformulated to be of the correct form. Consider, for instance, the problem of finding the generating function for binomial coefficients $\binom{n}{k}$ for fixed k as a function of n . The trick is to think of each set as being listed in order, from 1 to n . Take a set B with $2k+1$ points. There is a bijection between multi-indices on B and subsets with k elements. The multi-index corresponding to a subset counts the number of elements between each of the elements of the subset and also counts each element. So the ordinary generating function is

$$\begin{aligned} B_k(y) &= \sum_{n=0}^{\infty} \binom{n}{k} y^n \\ &= (1+y+y^2+y^3+\dots)y \cdots (1+y+y^2+y^3+\dots)y(1+y+y^2+y^3+\dots) \\ &= \frac{y^n}{(1-y)^{n+1}}. \end{aligned} \quad (8)$$

1.1.4 Integer partitions

The problem of counting integer partitions may also be studied by this method. Each integer partition corresponds to a type vector, which is a multi-index on the set of strictly positive natural numbers. Say that the type-vector is $p_1, p_1, p_3 \dots, p_j, \dots$, eventually zero. This determines another multi-index of the form $1p_1, 2p_2, 3p_3, \dots, jp_j, \dots$ whose values are multiples of j . The coefficient $p(k)$ of x^k is the number of such multi-indices that have sum k . So the ordinary generating function for integer partitions is

$$\begin{aligned} P(x) &= \sum_{k=0}^{\infty} p(k)x^k \\ &= (1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)(1+x^3+x^6+x^9+\dots)\dots \\ &= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots \end{aligned} \tag{9}$$

Let $p(k, [n])$ be the number of integer partitions of k into parts each of some size j with $j \leq n$. The ordinary generating function is

$$\begin{aligned} P_{[n]}(x) &= \sum_{k=0}^{\infty} p(k, [n])x^k \\ &= (1+x+x^2+x^3+\dots)(1+x^2+x^4+x^6+\dots)\dots(1+x^n+x^{2n}+x^{3n}+\dots) \\ &= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots(1-x^n)^{-1}. \end{aligned} \tag{10}$$

How about the seemingly more complicated problem of counting integer partitions of k into n parts? It may be proved by conjugation that the number $p(k, \leq n)$ of partitions into at most n parts is given by $p(k, \leq n) = p(k, [n])$. Therefore the ordinary generating functions are equal: $P_{\leq n}(x) = P_{[n]}(x)$. The ordinary generating function for integer partitions of k into n parts is then obtained by $p(k, n) = p(k, \leq n) - p(k, \leq n-1)$, giving

$$\begin{aligned} P_n(x) &= P_{\leq n}(x) - P_{\leq n-1}(x) = P_{\leq n-1}(x) \left[\frac{1}{1-x^n} - 1 \right] \\ &= P_{\leq n-1}(x) \frac{x^n}{1-x^n} = x^n P_{\leq n}(x). \end{aligned} \tag{11}$$

Notice that this immediately implies that $p(k, n) = p(k-n, \leq n)$.

1.1.5 Words and sentences

Here are applications of ordinary generating functions that go beyond multi-indices. Consider an alphabet with m letters. The number of words of length k is m^k . This is of course just the number of functions from a k element set to an m element set. The ordinary generating function is

$$(1-mx)^{-1} = \sum_{k=0}^{\infty} (mx)^k = \sum_{k=0}^{\infty} m^k x^k. \tag{12}$$

Now consider sentences of n words having a total of k letters from an alphabet of m letters. A sentence is a sequence of words, so the ordinary generating function is

$$(1 - mx)^{-n} = \sum_{k=0}^{\infty} \binom{n}{k} (mx)^k = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} m^k x^k. \quad (13)$$

This has a simple interpretation. There are m^k sequences of letters. These can be divided up into words by placing $n-1$ barriers between sequences, thus designating words.

1.1.6 Set partitions

There is an interesting ordinary generating function for set partitions of a set with k elements into n blocks. The number of such set partitions is the Stirling number of the second kind $S(k, n)$, so this is another way of getting at Stirling numbers of the second kind. The trick is to take the k element set that is being partitioned in order, from 1 to k . In each block of the set partition, consider the smallest element. This determines an ordering of the blocks. Consider an alphabet with n letters arranged in order. Then the set partition determines a sentence. The j th word in the sentence consists of the j th letter in the alphabet, followed by the other letters that occur in that block. The j th word only uses the first j letters of the n letters in the alphabet. The ordinary generating function for such sentences is

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{\infty} S(k, n) x^k \\ &= x(1-x)^{-1} x(1-2x)^{-1} x(1-3x)^{-1} \cdots x(1-nx)^{-1} \\ &= x^n (1-x)^{-1} (1-2x)^{-1} (1-3x)^{-1} \cdots (1-nx)^{-1}. \end{aligned} \quad (14)$$

The ordinary generating function for set partition numbers depends on an artificial ordering of the set. For such problems involving sets another tool is more natural: the exponential generating function.

1.2 Two variable

1.2.1 Binomial coefficients

There is something awkward about having two generating functions for $\binom{n}{k}$. Using two variables gives a nicer way to think of the situation. Thus consider

$$B(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} x^k y^n = \sum_{n=0}^{\infty} (1+x)^n y^n = \frac{1}{1-(1+x)y}. \quad (15)$$

This is a beautiful packaging of the facts about binomial coefficients. One has an experiment with many stages. At each stage one gets a 0 or 1 according to

whether there is a failure or a success at that stage. This is indicated by the $1 + x$ factor. Furthermore, in either case one counts this stage with a y factor. So at each stage there is a $(1 + x)y$ term. The infinite sequence of stages is then given by taking the geometric series $1/(1 - z)$ and replacing z by $(1 + x)y$.

One byproduct of this analysis is the other series for the binomial coefficients. One writes

$$B(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{n}{k} y^n x^k = \frac{1}{1-y} \frac{1}{1-\frac{y}{1-y}x} = \sum_{k=0}^{\infty} \frac{y^k}{(1-y)^{k+1}} x^k. \quad (16)$$

We recover the result

$$B_k(y) = \sum_{n=0}^{\infty} \binom{n}{k} y^n = \frac{y^k}{(1-y)^{k+1}}. \quad (17)$$

1.2.2 Multiset coefficients

One can make a similar calculation for the multiset coefficients $\left(\binom{n}{k}\right)$. Thus consider

$$M(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\binom{n}{k}\right) x^k y^n = \sum_{n=0}^{\infty} (1-x)^{-n} y^n = \frac{1}{1-\frac{1}{1-x}y}. \quad (18)$$

At each stage one gets a natural number. This is indicated by the $1/(1-x)$ factor. Furthermore, one counts this stage with a y factor. So at each stage there is a $y/(1-x)$ term. The infinite sequence of stages is then given by taking the geometric series $1/(1-z)$ and replacing z by $y/(1-x)$.

This should give the other series for the multiset coefficients. One writes

$$M(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\binom{n}{k}\right) y^n x^k = (1-x) \frac{1}{1-y} \frac{1}{1-\frac{1}{1-y}x} = (1-x) \sum_{k=0}^{\infty} \frac{1}{(1-y)^{k+1}} x^k. \quad (19)$$

After a calculation we get

$$M(x, y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\binom{n}{k}\right) y^n x^k = \frac{1}{1-y} + \sum_{k=1}^{\infty} \frac{y}{(1-y)^{k+1}} x^k. \quad (20)$$

Hence $M_0(y) = 1/(1-y)$ and $M_k(y) = y/(1-y)^{k+1}$ for $k \geq 1$.

1.2.3 Integer partition numbers

There is a double series for the integer partition numbers $p(k, n)$. This is

$$P(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p(k, n) x^k y^n = (1-yx)^{-1} (1-yx^2)^{-1} (1-yx^3)^{-1} \dots \quad (21)$$

To check this, one can expand

$$(1 - yx^j)^{-1} = \sum_{p=0}^{\infty} y^p x^{jp}. \quad (22)$$

Take the product of such factors. The coefficient of the $x^k y^n$ term is the number of type vectors p_1, p_2, p_3 with $p_1 + p_2 + p_3 + \dots = n$ and $p_1 + 2p_2 + 3p_3 + \dots = k$. This number is $p(k, n)$. The x acts as a counter for k , while the y acts as a counter for n .

2 Exponential generating functions

2.1 One variable

2.1.1 Generalities

An exponential generating function is a convenient way of working with a sequence of numbers a_k defined for $k \geq 0$. It is a power series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}. \quad (23)$$

The utility of exponential generating functions is that they are relevant for combinatorial problems involving sets and subsets. Often there is a set A , and one wants to consider functions from A to another set B . For each such function there is a decomposition of A into subsets indexed by B . Another scenario is when there is only the set A and one is interested in set partitions of A . It is the sets that are of primary interest in constructing the combinatorial objects. The number of combinatorial objects may depend only on the number of elements of the set, but the way combinatorial objects are generated is via decompositions of sets into subsets, rather than decompositions of numbers into summands.

Say that one has a collection of functions $f^{(j)}(x)$ for j in some index set B . For notational simplicity we take B to be the set $[n]$ of natural numbers from 1 to n . Suppose also that

$$f^{(j)}(x) = \sum_{k=0}^{\infty} a_k^{(j)} \frac{x^k}{k!}. \quad (24)$$

Let

$$f(x) = f^{(1)}(x)f^{(2)}(x)\dots f^{(n)}(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}. \quad (25)$$

be the product of the exponential generating functions. Then the coefficient a_k corresponding to the product is

$$a_k = \sum_{k_1+k_2+\dots+k_n=k} \frac{k!}{k_1! \dots k_n!} a_{k_1}^{(1)} a_{k_2}^{(2)} \dots a_{k_n}^{(n)}. \quad (26)$$

This is a sum over all multi-indices k_1, \dots, k_n with $k_1 + k_2 + \dots + k_n = k$ of combinatorial coefficients satisfying a product rule. However there is also a multinomial coefficient. This counts the number of functions from a k element set A_k to an n element set $B = [n]$ with the given multi-index. Such a function corresponds to a decomposition of A into subsets, indexed by $B = [n]$. We think of each subset having a corresponding combinatorial coefficient $a_{k_j}^{(j)}$ that depends only on j and on the size k_j of this subset. So we can write instead

$$a_k = \sum_{[n]^{A_k}} a_{k_1}^{(1)} a_{k_2}^{(2)} \cdots a_{k_n}^{(n)}. \quad (27)$$

Here A_k is a set with k elements, and $B = [n]$ is a set with n elements, which we may think of as the numbers from 1 to n . The set $[n]^{A_k}$ consists of all functions from A_k to $[n]$, that is, of all decompositions of A into subsets indexed by $[n]$. Then $a_{k_j}^{(j)}$ is the combinatorial coefficient associated with the subset of A of size k_j that corresponds to element j of $[n]$. So a complicated operation involving a huge sum over indexed decompositions of a set corresponds to a simple operation on the exponential generating function: multiplication.

One use for exponential generating functions is to count indexed decompositions into subsets of a set A . The subsets are indexed by another set B . Say that A is a display table with k places for placing fruits. The set B of categories of fruits consists of apple, orange, and banana. Say that one wants to know how many ways there are to place up to 2 apples, 3 or 4 oranges, and an even number of bananas in the k locations. The exponential generating function is $f(x) = (1 + x + \frac{1}{2}x^2)(\frac{1}{6}x^3 + \frac{1}{24}x^4) \cosh(x)$. The factors correspond to the different kinds of fruits; the terms within the factors give the inventory of available subsets of locations on the table for each kind of fruit. A solution is a choice of a subset for each kind of fruit. Alternatively, it is a function that assigns to each location on the table a corresponding fruit.

Here is a variation on the idea. Say that one has a function

$$f(x) = \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}. \quad (28)$$

Let

$$g(x) = \frac{1}{n!} f(x)^n = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}. \quad (29)$$

Then the coefficient b_k is

$$b_k = \sum_{k_1+k_2+\dots+k_n=k} \frac{1}{n!} \frac{k!}{k_1! \cdots k_n!} a_{k_1} a_{k_2} \cdots a_{k_n}. \quad (30)$$

This is a sum over all multi-indices k_1, \dots, k_n with each $k_j \geq 1$ and with $k_1 + k_2 + \dots + k_n = k$ of combinatorial coefficients satisfying a product rule. There is also a multinomial coefficient. This counts the number of surjections from a k

element set A_k to an n element set with the given multi-index. Such a function corresponds to a decomposition of A into subsets, indexed by $[n]$. We think of each subset having a corresponding combinatorial coefficient a_{k_j} that depends only on the size k_j of this subset. On the other hand, we also have a factor of $1/n!$, which means that we are counting set partitions rather than surjections. So we can write instead

$$b_k = \sum_{\text{Part}_n(A_k)} a_{k_1} a_{k_2} \cdots a_{k_n}. \quad (31)$$

Here A_k is a set with k elements, and $\text{Part}_n(A_k)$ consists of all set partitions of A_k into n blocks, that is, into all decompositions of A_k into n blocks, disregarding order. Then a_{k_j} is the combinatorial coefficient associated with a block in the set partition of A of size k_j . So a complicated operation involving a huge sum over set partitions of a set into n blocks corresponds to a simple operation on the exponential generating function: $1/n!$ times the n th power.

Here is an illustration. Say that A is a display table with k places. One wishes to use these places for displaying 4 formal table settings that each have an odd number of pieces. A preliminary problem is then to partition A into 4 subsets each of which has an odd number of elements. The exponential generating function is $g(x) = \frac{1}{4!} \sinh^4(x)$. Consider for example the special case $k = 10$. It is not hard to compute the number directly. There are three integer partitions of 10 into odd parts. We can partition the set into blocks of sizes 3,3,3,1 in 2800 ways, partition the set into blocks of sizes 5,5 in 2520 ways, and partition the set into blocks of sizes 7,1,1,1 in 120 ways. The total is 5440. But it is perhaps easier to get this from the exponential generating function. An elementary computation gives

$$g(x) = \frac{1}{4!} \sinh^4(x) = \frac{1}{4} \left[\frac{1}{8} \sinh(4x) - \frac{1}{2} \sinh(2x) + \frac{3}{8} \right]. \quad (32)$$

So the coefficient of $\frac{1}{10!}x^{10}$ in $g(x)$ is $\frac{1}{4!}[\frac{1}{8}4^{10} - \frac{1}{2}2^{10}]$ which is again 5440.

Here is yet another step. Say that one has a function

$$f(x) = \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}. \quad (33)$$

Let

$$h(x) = e^{f(x)} = \sum_{k=0}^{\infty} c_k \frac{x^k}{k!}. \quad (34)$$

Notice that

$$e^{f(x)} = \sum_{n=0}^{\infty} \frac{f(x)^n}{n!}. \quad (35)$$

So we are dealing now with set partitions into an arbitrary number of blocks. It follows that

$$c_k = \sum_{\text{Part}(A_k)} a_{k_1} a_{k_2} \cdots a_{k_n}. \quad (36)$$

Here A_k is a set with k elements, and $\text{Part}(A_k)$ consists of all set partitions of A_k into blocks, that is, into all decompositions of A_k into blocks, disregarding order. The number n of blocks is a variable depending on the set partition. Then a_{k_j} is the combinatorial coefficient associated with a block in the set partition of A . So a complicated operation involving a huge sum over set partitions of a set corresponds to a simple operation on the exponential generating function: taking the exponential.

Here is an illustration. Say that A is a display table with k places. One wishes to use these places for displaying formal table settings that each have an odd number of pieces. Partition A into blocks each of which has an odd number of elements. The exponential generating function is $h(x) = e^{\sinh(x)}$.

2.1.2 Functions

Here are some important examples. An exponential generating function for the functions from a k -set to an n set is

$$(e^x)^n = e^{nx} = \sum_{k=0}^{\infty} n^k \frac{x^k}{k!}. \quad (37)$$

This follows from the product property, since a function from a k -set to an n set is just a decomposition of the k -set into subsets indexed by the n -set.

2.1.3 Injections

An exponential generating function for the injective functions from a k -set to an n set is

$$(1+x)^n = \sum_{k=0}^{\infty} (n)_k \frac{x^k}{k!}. \quad (38)$$

This follows from the product property, since an injective function from a k -set to an n set is just a decomposition of the k -set into subsets with at most one element indexed by the n -set.

Another exponential generating function for the injective functions from a k set to an n set is

$$y^k e^y = \sum_{m=0}^{\infty} \frac{y^{m+k}}{m!} = \sum_{n=0}^{\infty} (n)_k \frac{y^n}{n!}. \quad (39)$$

This also follows from the product property, since an injective function from a k -set to an n set is obtained by taking a sequence of k one-point subsets of the n -set together with a remaining subset.

2.1.4 Surjections and set partitions

The exponential generating function for surjections from a k -set to an n set is

$$(e^x - 1)^n = \sum_{k=0}^{\infty} n! S(k, n) \frac{x^k}{k!}. \quad (40)$$

A surjective function is the same as a decomposition into a sequence of n non-empty subsets. The exponential generating function for set partitions of a k -set into n blocks is

$$\frac{1}{n!}(e^x - 1)^n = \sum_{k=0}^{\infty} S(k, n) \frac{x^k}{k!}. \quad (41)$$

The exponential generating function for set partitions of a k -set is

$$e^{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \quad (42)$$

2.1.5 Permutations

If A is a set, then a permutation of A is a bijection from A to itself. If A has k elements, then the number of permutations is $k!$. For every permutation of A there is a set partition of A into subsets, on each of which there is a cyclic permutation. A cyclic permutation is a permutation that consists of a single cycle.

The number of cyclic permutations of a set with k elements is $(k - 1)!$. So the exponential generating function for cyclic permutations is

$$\sum_{k=0}^{\infty} (k - 1)! \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k} x^k = -\log(1 - x). \quad (43)$$

This is consistent with what we know about permutations. Permutations are partitioned into cyclic permutations. Therefore the exponential generating function for permutations is obtained by taking the exponential of that for cyclic permutations:

$$e^{-\log(1-x)} = \frac{1}{1-x} = \sum_{k=0}^{\infty} k! \frac{x^k}{k!} \quad (44)$$

Now let $c(k, n)$ be the number of permutations of a k element set with n cycles. The exponential generating function is

$$\sum_{k=0}^{\infty} c(k, n) \frac{x^k}{k!} = \frac{1}{n!} (-\log(1 - x))^n. \quad (45)$$

2.2 Two variable

2.2.1 Functions

The two-variable exponential generating function for the functions from a k -set to an n set is

$$e^{e^x y} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} n^k \frac{x^k}{k!} \frac{y^n}{n!}. \quad (46)$$

A function from A to B is given by a set partition of the disjoint union $A + B$ into blocks that consist of a single element of B together with a subset of A .

2.2.2 Injections

The two-variable generating function for the injective functions from a k -set to an n set is

$$e^{(1+x)y} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} \frac{x^k}{k!} \frac{y^n}{n!}. \quad (47)$$

A injective function from A to B is given by a set partition of the disjoint union $A + B$ into blocks that consist of a single element of B together with a subset of A with at most one element.

2.2.3 Surjections and set partitions

The exponential generating function for surjections from a k -set to an n set is

$$e^{(e^x-1)y} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} n! S(k, n) \frac{x^k}{k!} \frac{y^n}{n!}. \quad (48)$$

A surjective function from A to B is given by a set partition of the disjoint union $A + B$ into blocks that consist of a single element of B together with a non-empty subset of A .

The exponential generating function for set partitions of a k -set is

$$e^{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \quad (49)$$

Write

$$e^{e^x y} = e^{(e^x-1)y} e^y \quad (50)$$

This says that a function from A to B is equivalent to decomposing $A + B$ into two sets, and partitioning each of these sets. The first set is partitioned into blocks that consist of a single element of B together with a non-empty subset of A . This is the surjection component. The second set is partitioned into the blocks that consist of a single element of B alone. For fixed k this gives

$$\sum_{n=0}^{\infty} n^k \frac{y^n}{n!} = \left(\sum_{n=0}^{\infty} S(k, n) y^n \right) e^y. \quad (51)$$

This says that a function from A to B is obtained by decomposing B into the range of the function and its complement. For fixed n this gives the remarkable identity

$$n^k = \sum_{m=0}^{\infty} S(k, m) (n)_m. \quad (52)$$

2.2.4 Permutations

Now let $n!c(k, n)$ be the number of permutations of a k element set with n cycles, where the cycles are labeled. The exponential generating function is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} n!c(k, n) \frac{x^k}{k!} \frac{y^n}{n!} = e^{-\log(1-x)y} = \left(\frac{1}{1-x} \right)^y. \quad (53)$$

This may be expanded by the binomial theorem as

$$(1-x)^{-y} = \sum_{k=0}^{\infty} \binom{-y}{k} (-x)^k = \sum_{k=0}^{\infty} (-y)_k (-1)^k \frac{x^k}{k!}. \quad (54)$$

This shows that

$$\sum_{n=0}^{\infty} n!c(k, n) \frac{y^n}{n!} = (-y)_k (-1)^k = (y)^{(k)}. \quad (55)$$

Here $(y)^{(k)}$ is the rising factorial. For instance, $(y)^{(2)} = y(y+1) = y^2 + y$, while $(y)^{(3)} = y(y+1)(y+2) = y^3 + 3y^2 + 2y$, and $(y)^{(4)} = y(y+1)(y+2)(y+3) = y^4 + 6y^3 + 11y^2 + 6y$. These polynomials give a convenient way of counting permutations with a given number of cycles. For instance, for permutations of a set with 4 elements there are 6 permutations with one cycle, 11 permutations with two cycles, 6 permutations with three cycles, and 1 permutation with four cycles.