review of Quantum Fields and Strings: A Course for Mathematicians

reviewed by William Faris, Department of Mathematics, University of Arizona

email: faris@math.arizona.edu

version 2 of review

Quantum Fields and Strings: A Course for Mathematicians, Volumes 1 and 2

edited by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten

American Mathematical Society, Providence, Rhode Island, 1999.

Introduction.

These two volumes of roughly 1500 pages contain the lecture notes for courses given during the 1996–1997 Special Year in Quantum Field Theory held at the Institute for Advanced Study in Princeton. The stated goal was to "create and convey an understanding, in terms congenial to mathematicians, of some fundamental notions of physics, such as quantum field theory, supersymmetry and string theory," with emphasis on the intuition stemming from functional integrals. The motivation for this effort was the recent impact of quantum field theory on the formulation of new conjectures and concepts in geometry and algebra. In many cases mathematicians have been able to verify these conjectures, but the proofs have dealt with each individual case and ignore the bigger picture that governs the physicists' intuitions. This series of courses was intended to teach mathematicians the physical concepts that underlie these conjectures and concepts. Since this is frontier physics, this is not an easy task, and consequently there is considerable diversity in mathematical rigor among the courses recorded in these volumes. Nevertheless, it is remarkable to see prominent mathematicians working so hard as students of a new subject. They even did homework and took exams. We shall discuss their level of success at the end of this review.

The first volume consists of Parts 1 and 2, while the second volume consists of Parts 3 and 4. Each part consists of several texts or lecture series together with shorter technical contributions on special topics. Here is a list of the contents of each part:

Part 1: Classical Fields and Supersymmetry.

Pierre Deligne and John W. Morgan, Notes on Supersymmetry

Pierre Deligne, Notes on Spinors

Pierre Deligne and Daniel S. Freed, Classical Field Theory + Supersolutions + Sign Manifesto

Part 2: Formal Aspects of QFT.

Pierre Deligne, Notes on Quantization

David Kazhdan, Introduction to QFT

Edward Witten, Perturbative Quantum Field Theory + Index of Dirac Operators

Ludwig Fadeev, Elementary Introduction to Quantum Field Theory

David Gross, Renormalization Groups

Pavel Etingof, Note on Dimensional Regularization

Edward Witten, Homework

Part 3: Conformal Field Theory and Strings

Krzysztof Gawędzki, Lectures on Conformal Field Theory

Eric D'Hoker, String Theory

Pierre Deligne, Super Space Descriptions of Super Gravity

Dennis Gaitsgory, Notes on $2d$ Conformal Field Theory and String Theory

Andrew Strominger, Kaluza-Klein Compactifications, Supersymmetry, and Calabi-Yau Spaces

Part 4: Dynamical Aspects of QFT

Edward Witten, Dynamics of Quantum Field Theory

Nathan Seiberg, Dynamics of $N = 1$ Supersymmetric Field Theories in Four Dimensions

Dante described nine circles of Hell, at deeper and deeper levels. This review follows his scheme, describing nine levels of the architecture of field theory.

Level 1. Relativity.

Relativity here means Einstein's special relativity. The geometrical formulation of this theory is the space-time of Minkowski. It is convenient for many purposes to consider space-time of arbitrary dimension n. A *field* is a function from space-time to some other space (of scalars, vectors, etc.).

In relativity it is convenient to take time and space to have the same units. But there is a deeper relation between time and space that allows a transition between *Minkowski* and *Euclidean* fields. A Minkowski field is a function defined on n dimensional relativistic space-time. A natural example is a solution of the wave equation. A Euclidean field is a function defined on n dimensional Euclidean space. The corresponding example is a solution of the Laplace equation. One of the Euclidean coordinates is an imaginary time coordinate, and a process of analytic continuation leads back and forth between Minkowski and Euclidean fields. This has the technical name *Wick rotation*.

A *string* is a function from a two-dimensional surface to space-time. This is something like a field defined on a domain of dimension two, but in this theory it is the values of the function that are points in space-time. Also, in the context of string theory it is common to consider a more general notion of space-time. For instance it could be the product of a Minkowski space-time with a compact Riemannian manifold.

Level 2. Quantum theory.

Quantum mechanics is analogous to probability. In probability the current state of a random system can be given by a *probability density*, which is a function with positive values and integral one. Say there is such a probability density at time zero. The time evolution is defined by a positive *transition probability* function that determines for each t and each initial point a conditional probability density for the final point. This in turn determines the probability density at time t. At every stage of the calculation there are only positive numbers. Compare this with quantum mechanics. The current state of a quantum system is determined by a *state function* with complex values. Say there is such a state function at time zero. The time evolution is defined by a *transition amplitude* function. This function depends on the time t and on the initial and final points and has complex values. This determines the state function at time t. Complex numbers occur at every stage of the quantum mechanical calculation, except at the end. Then the absolute value of the final state function, when squared, is interpreted as a probability density, from which experimental predictions are made.

The transition functions in probability and in quantum mechanics may be calculated by the use of *path integrals*. In probability the transition probability for going from one point to another is given by taking an integral over all paths that start from the initial point and travel for a time t. This is an integral with respect to a probability measure, with a constraint that determines the final point. In quantum mechanics the amplitude for going from one point to another is also given by an integral over all paths that go from the initial point to the final point in time t. However the numbers that occur in the quantum mechanical path integral are complex numbers.

The scale on which quantum mechanics becomes important is measured by Planck's constant. This constant is analogous to a variance parameter in probability. With units in which Planck's constant is one the unit of energy is the same as the unit of frequency. This is often combined with the unit correspondences coming from relativity, where mass and energy have the same units. As a consequence, the same regime may be described as one of short distance, short time, ultraviolet frequency, high energy, or high mass. The opposite regime is that of large distance, long time, infrared frequency, low energy, or low mass. The same coefficient can define the frequency of an oscillator, the time decay rate of a process, the spatial decay profile of an interaction, or the mass of a particle.

Level 3. Fields.

Quantum field theory is the currently accepted theory of the elementary particles and their interactions. For instance, quarks (the constituents of protons and neutrons) and electrons are described by quantum fields. The interactions (electromagnetic and nuclear forces) between these particles are also described by quantum fields. In principle, leaving aside gravitation, quantum field theory is a candidate for the ultimate theory of all matter and force.

A classical field is a function defined on space-time whose values are scalars or vectors or some other geometrical objects. A quantum field, then, should be a field described by quantum mechanics rather than by classical mechanics. If the world is described by quantum fields, then one would think that the measurement of such fields would be a commonplace in the physics laboratory. In fact, this is not the case, and the physical interpretation of a quantum field is quite indirect. According to the scattering theory of quantum fields, at large times a quantum field in a particular state is asymptotic to a free quantum field, that is, a quantum field that obeys a linear wave equation. Furthermore, a free quantum field has a complementary interpretation as a theory of a system of N identical particles. In fact, all possible numbers $N = 0, 1, 2, 3, \ldots$ of particles are described by the one quantum field. Most laboratory observations in elementary particle physics are interpreted as recording the passage of particles. This is quite strange: the fundamental equations are written in terms of fields, but the interpretation is in terms of particles. In addition to these difficult questions of theory and experiment, there is also the problem that the theory itself is not in finished form.

Why should a mathematician care about this subject? One obvious answer is that it is intrinsically interesting to look at the most fundamental physical theories. Anyone with curiosity about the world might want to know the principles describing matter and motion. If one also knows some mathematics, then it is possible to get at least some idea of the structure that is being proposed by quantum field theory. This, at least ideally, should bring pleasure.

However, there is another reason for a mathematician to be excited about this theory: it impacts several areas of mathematics. One impact is on geometry, and this was motivation for the course at Princeton that led to the volumes under review. Physicists have used quantum field theory and string theory to produce new conjectures and concepts in geometry. In many cases, mathematicians have been able to verify the conjectures, but it is striking to see physics lead mathematics in this way. Another impact is on certain areas of probability (such as random fields) and on the related field of statistical mechanics. Here again it has turned out that a rather abstruse quantum field theory formulation can point the way to new conjectures and to overall conceptual unity. Even if quantum field theory turns out to be totally wrong physically, its apparatus will remain part of mathematics.

The Wick rotation correspondence is important in modern work on quantum field theory. The analog of a *quantum field* on Minkowski space-time is a *random field* on Euclidean space. In Minkowski space the positive linear functional that defines moments of the quantum fields is called the *vacuum state*. The corresponding moments as functions of space-time points are called *Wightman functions*. In Euclidean space the positive linear functional that defines moments of the corresponding fields is an *expectation* in the usual sense of probability. The corresponding moments as functions of space points are called *Schwinger* functions. A theorem of Osterwalder and Schrader says that the Wightman functions can be recovered from the Schwinger functions. Quantum field theory is reduced to random field theory, which is a part of probability.

Another important distinction is between *free field* and *interacting field*. A free field is a field that obeys a linear wave equation. An interaction is a non-linearity in the field equation. These concepts may also be expressed in terms of action (in the Minkowski case) or energy (in the Euclidean case). In the free field case the action or energy is quadratic. A quadratic energy in the Euclidean case corresponds to a Gaussian random field.

Even though physics may require something else, in order to get an idea of the difficulties of quantum field theory it is nicest to look at the case of a scalar field. By the Osterwalder-Schrader theorem one can consider a random field defined on Euclidean space. Here is the general pattern. Consider real functions ϕ on n dimensional Euclidean space. The quantity that underlies the theory is the Euclidean action (or energy functional)

$$S_E(\phi) = \int \left(\frac{1}{2} |\nabla_x \phi|^2 + p(\phi) \right) \, d^n x, \tag{1}$$

where $d^n x$ is Lebesgue measure on n dimensional Euclidean space. The interaction p can be taken to be a polynomial of even degree with positive leading coefficient. The corresponding classical equation is obtained by setting the variational derivative equal to zero. Thus it is

$$S'_E(\phi) = -\nabla_x^2 \phi + p'(\phi) = 0. \tag{2}$$

The random field is given formally by specifying its probability measure. This measure is supposed to be given by

$$d\mu_E(\phi) = \frac{1}{Z} e^{-S_E(\phi)} \prod_x d\phi(x). \tag{3}$$

The quantity Z is a normalization constant. Of course this is all highly non-rigorous, since there is no Lebesgue measure on infinite dimensional space.

When the interaction is quadratic a rigorous construction is straightforward. A specification of mean and covariance functions always determines a unique Gaussian measure. In the field theory case the interaction that gives a Gaussian free field is

$$p(\phi) = \frac{1}{2} m^2 \phi^2, \tag{4}$$

where $m \geq 0$ is a mass parameter. The Euclidean action is then

$$S_E(\phi) = \frac{1}{2} \langle \phi, A\phi \rangle, \tag{5}$$

where A is the differential operator

$$A = -\nabla_x^2 + m^2. \tag{6}$$

The derivative of the action is a linear differential operator $S'(\phi) = A\phi$. Such differential operators act on spaces of functions that have decay properties at infinity. It turns out that when $m > 0$ or the dimension $n > 2$ the homogeneous equation $A\phi = 0$ has only the zero solution, and there is a natural definition of the inverse operator $C = A^{-1}$. The Gaussian field has mean zero and covariance

$$C = A^{-1} = (-\nabla_x^2 + m^2)^{-1}. \tag{7}$$

The reason for the restriction on dimension is that finding this inverse is equivalent to solving an equilibrium problem $A\phi = \chi$ with a source χ. When $m > 0$ this is a problem of diffusion and dissipation, and there is always an equilibrium. When $n > 2$ and $m = 0$ it is a problem of pure diffusion. But there is enough room in space for the material produced by the source to diffuse away. On the other hand, when $m = 0$ and $n \leq 2$ the material produced by the source just keeps accumulating, and there is no equilibrium. The parameter m

has the interpretation of a mass or a frequency. The problem with $m = 0$ and $n \leq 2$ is a problem of *infrared* frequency, or long distance.

The covariance operator C may be given explicitly as an integral operator involving a function of two variables (the Green's function) that is the analog of a matrix. Thus, for example, when $n > 2$ the covariance has the asymptotic form

$$C(x,y) = (-\nabla_x^2 + m^2)^{-1}(x,y) \sim \frac{1}{(n-2)a_n} \frac{1}{|x-y|^{n-2}} e^{-m|x-y|} \tag{8}$$

for $|x - y|$ large. In this formula a_n is the area of the $n-1$ dimensional sphere S_{n-1} in n dimensional Euclidean space. When $n \geq 2$ the covariance function has singularities on the diagonal $x = y$, and this turns out to have profound consequences for the theory. In particular, this says that for $n \geq 2$ the variance of the field $\phi(x)$ is infinite at each point. We shall see that this implies that the fields must be regarded as generalized functions. (In this review we take generalized function to be a synonym for tempered Schwartz distribution.) Only the smeared out fields

$$\phi(f) = \int \phi(x) f(x)\, d^n x \tag{9}$$

are meaningful as Gaussian random variables. They have mean zero and covariance

$$\mathbf{E}[\phi(f)\phi(g)] = \int\int f(x)(-\nabla_x^2 + m^2)^{-1}(x,y)g(y)\, d^n x\, d^n y, \tag{10}$$

and these are finite. In conclusion, in the Gaussian case the formal measures on the space of generalized functions are well-defined probability measures.

The problem with the local singularities for $n \geq 2$ is a problem with *ultraviolet* frequency or short distance. There is no ultraviolet problem in the case of a Euclidean space of dimension $n = 1$. Both the free and interacting fields are well-defined random functions. The coordinate x is space or imaginary time. The free field covariance is given by the Green's function

$$C(x,y) = (-\frac{d^2}{dx^2} + m^2)^{-1}(x,y) = \frac{1}{2m} e^{-m|x-y|}. \tag{11}$$

The corresponding free random field is a Gaussian random field with mean zero and covariance given by this Green's function. Thus $\phi(x)$ is a random variable for each x, and the covariances of these random variables are given by

$$\mathbf{E}[\phi(x)\phi(y)] = C(x,y). \tag{12}$$

This is a famous stochastic process sometimes known as the stationary Ornstein-Uhlenbeck process. A typical sample path is a function that generally remains close to zero, straying by an amount measured by the variance $C(0,0) = 1/(2m)$. When it fluctuates much above this, it tends to return to near zero in an interval of length equal to several multiples of the correlation length $1/m$.

This formal expression for the measure would seem to indicate that the measure assigns probability one to functions ϕ such that $S_E(\phi)$ is finite. This is not the case. The space of functions such that $S_E(\phi)$ is finite is a Hilbert space. There is a general theory that says that if one has a Gaussian measure associated with such a Hilbert space structure, then the measure is concentrated on a Banach space, but this space must be taken somewhat larger than the Hilbert space. In the present case, for instance, it could be taken to be a Banach space of continuous functions satisfying a growth condition at infinity. That is, the fields will be continuous with probability one, but also nowhere differentiable with probability one.

The interacting field in dimension $n = 1$ is also tractable. The action is

$$S_E(\phi) = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + p(\phi) - E \right) dx. \tag{13}$$

The constant E is arbitrary; a shift in E corresponds to an (infinite) renormalization of the constant Z in the corresponding measure. There is a remarkable construction of this measure. The idea is to replace the interaction p, which is used to suppress the field, with a drift u, which is used to shift it. To find u, consider the eigenvalue equation

$$-\frac{1}{2}F''(z) + p(z)F(z) = EF(z). \tag{14}$$

Take E to be the smallest eigenvalue and $F(z) > 0$ the corresponding eigenfunction. Let $H = \log F$. Then $u = H'$. The relation between u and p is given by

$$u' + u^2 = 2(p - E). \tag{15}$$

This is a curious nonlinear relation; its deeper significance will appear in the discussion of supersymmetry. Explicit computations are usually difficult, except for the case where the $p(z) = (1/2)m^2z^2$, which leads to $H(z) = -(1/2)mz^2$ and a drift term $u(z) = -mz$.

White noise is the generalized Gaussian stochastic process $\alpha(x)$ with mean zero and with covariance given by the identity operator. Thus

$$\mathbf{E}[\alpha(x)\alpha(x')] = \delta(x - x'). \tag{16}$$

There is a unique global solution $\phi(x)$ of the nonlinear stochastic differential equation

$$\frac{d\phi(x)}{dx} - u(\phi(x)) = \alpha(x) \tag{17}$$

that does not grow exponentially. The nonlinear mapping from α to ϕ maps the Gaussian white noise probability measure to a probability measure on fields ϕ. The relation between u and p may be used to show that $\phi(x)$ is the desired interacting random field.

The more formidable problem is with an interacting field in dimension $n \geq 2$. This should be a probability measure on the space of generalized functions. The expression for the action involves higher powers such as $\phi(x)^k$ for $k > 2$. If the field were a function, this would present no problem. But defining a power of a generalized function is a more delicate matter. The construction of the non-Gaussian probability measure that defines an interacting field is a highly non-trivial enterprise.

Does such a field exist? The answer depends on the dimension. The method just described gives a construction for dimension $n = 1$. Glimm and Jaffe constructed interacting fields in dimensions $n = 2$ and $n = 3$ as part of a program of *constructive field theory*. See [1] and [2] for accounts of this enterprise. This work developed important new techniques, including rigorous *cluster expansions*. There is strong evidence [3] that for $n = 4$, the dimension of most obvious physical interest, the Bose field with self-interaction will not exist, except in the linear (Gaussian) case. This might be upsetting if this were the only kind of field. However, as we shall see below, there are other possibilities.

The quantum case is similar, but there are even more difficulties. Consider real functions ϕ on n dimensional Minkowski space with time and space coordinates $x = (t, \mathbf{x})$. The quantity that underlies the theory is the Minkowski action (or energy functional)

$$S_M(\phi) = \int \left(\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} |\nabla_{\mathbf{x}} \phi|^2 - p(\phi) \right) d^n x \tag{18}$$

with $d^n x = d^{n-1}\mathbf{x}\, dt$. As before, p is a polynomial of even degree with positive leading coefficient. The corresponding classical equation is obtained by setting the variational derivative equal to zero. Thus it is

$$S'_M(\phi) = -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2_{\mathbf{x}} \phi - p'(\phi) = 0. \tag{19}$$

Transition amplitudes for the quantum field are also given by a path integral, but this integral is given formally by a complex measure

$$d\mu_M(\phi) = \frac{1}{Z} e^{iS_M(\phi)} \prod_x d\phi(x). \tag{20}$$

In the free field case the Minkowski action is of the form

$$S_M(\phi) = \frac{1}{2}\langle \phi, A\phi \rangle, \tag{21}$$

where A is the differential operator

$$A = -\frac{\partial^2}{\partial t^2} + \nabla_{\mathbf{x}}^2 - m^2. \tag{22}$$

The derivative of the action is a linear differential operator $S'(\phi) = A\phi$. Now there is a problem. The equation $A\phi = 0$ is a classical wave equation, and it has many solutions in the space of generalized functions. So A cannot have an inverse. At most it can have a right inverse, and this right inverse is not unique. A choice of a right inverse corresponds to a choice of boundary conditions. When $m > 0$ or $d > 2$ it is possible to specify the inverse by analytic continuation from the Euclidean case, by the process of Wick rotation described before. The quantum analog of the covariance also has an extra factor of i. This analog is called the *propagator*, and it is determined by a limit

$$C = i \lim_{\epsilon \downarrow 0} \left(-\frac{\partial^2}{\partial t^2} + \nabla_{\mathbf{x}}^2 - m^2 + i\epsilon\right)^{-1}. \tag{23}$$

So it is possible to construct a free quantum field and to compute many quantities explicitly.

The situation for the interacting quantum field is quite different. All the difficulties of the Euclidean field are present, together with new headaches associated with the fact that the formal measure is oscillatory rather than positive. The Osterwalder-Schrader theorem enables, at least in principle, a construction of the quantum field from the Euclidean field by analytic continuation. However the Euclidean field obscures much of the physics, including the scattering behavior that is at the basis of much of the physical interpretation. This scattering behavior is only seen directly when working in the Minkowski space quantum field theory.

Kazhdan's lectures in Part 1 treat the axiomatic approach to quantum field theory. This is an attempt to prove general theorems about quantum fields that can be used in navigating the maze of specific examples. There is a logical chain. Imagine that one has been able to construct the Schwinger functions of an Euclidean field. These are moments of random variables. The Osterwalder-Schrader theory says there is an analytic continuation that gives the Wightman functions of a quantum field. The reconstruction theorem produces the quantum field itself. Finally, the Haag-Ruelle theory shows that the quantum field has good scattering behavior, and so it may be interpreted as quantum particles going in one direction and coming out in another direction.

For the axioms to have content, there must be interesting examples of random fields. While the constructive approach is not part of the book, the lectures in Part 2 by Witten and by Fadeev give an idea of the difficulties in dealing with these objects. The worst difficulty is the ultraviolet (short distance) *divergences*. The covariance functions for random fields or quantum fields have singularities, and these singularities are worse as the dimension gets higher. These singularities are supposed to be overcome by the process of *renormalization*. Imagine that the measure defining the field is approximated by a measure describing a field where the singularities have been smoothed out. This is the introduction of an ultraviolet cutoff. This approximation destroys many of the symmetries of the field, so it is not satisfactory in itself. However one can choose the parameters of the approximate measure so that certain physical quantities have reasonable values. Then the approximation is removed, and at the same time the parameters are changed in such a way that the physical quantities keep the same values. This is the renormalization. The parameters may become infinite, but one hopes that there is still a quantum field in the limiting case, and that it has the desired symmetries. This is the field that should describe Nature.

Level 4. Bose and Fermi fields.

The story so far is rather nice: quantum fields on Minkowski space correspond to random fields on Euclidean space. It turns out that this is an oversimplification. There are two kinds of quantum fields: *Bose* and *Fermi* fields. A Bose field describes identical particles that can occupy the same state, while a Fermi field describes identical particles that refuse to be in the same state. The fields described above are Bose fields. Elementary particles such as electrons and quarks are described by Fermi fields, and their behavior is heavily influenced by this exclusion principle. Again one can consider Fermi fields either in the Minkowski or the Euclidean versions. However even the Euclidean version of a Fermi field is an abstract algebraic structure involving anticommuting variables.

One would like to think of a Euclidean Fermi field as obtained by integrating the exponential of an action over a space of paths. What are these classical paths, and what is the integral? The answer is curious indeed. A classical Fermi field is not a function at all, but a non-commutative analog of a function. We can think of a typical function $F(\phi(x_1), \ldots, \phi(x_m))$ of a Bose field as being given by a polynomial in variables $\phi(x_1), \ldots, \phi(x_m)$. These variables commute, so the polynomials have the structure of a symmetric tensor algebra. On the other hand, a "function" $F(\psi(x_1), \ldots, \psi(x_m))$ of a Fermi field is, by definition, an element of an antisymmetric tensor algebra, also called a Grassmann algebra or an exterior algebra. That is, the generators $\psi(x_j)$ satisfy $\psi(x_j)\psi(x_k) = -\psi(x_k)\psi(x_j)$. The integral on this Grassman algebra is the linear functional that picks out the coefficient of the element of top degree. Of course this does not make sense in the infinite dimensional case. For Bose fields there is a similar problem, the lack of a notion of Lebesgue measure in infinite dimensional space. This was overcome by instead using Gaussian measures as a starting point. The same idea works for Fermi fields; there is a Gaussian functional analog that works in the infinite dimensional case.

The non-commutativity of the Euclidean Fermi fields means that one has left the domain of probability. However, the analogy with probability is strong, and in some ways the Fermi fields are nicer than the Bose fields. This is ultimately due to the relation $\psi(x_j)^2 = 0$, which expresses the exclusion principle in mathematical form.

Level 5. Nonabelian gauge fields.

One might take comfort in the nice probability picture for Bose fields. However this would be premature. The Bose fields that describe photons and other interactions are not scalar fields, or vector fields, or even tensor fields. They are *gauge fields*. This means that they do not describe quantities associated with individual points, but instead they describe how to transport these quantities along curves. Thus, for instance, the magnetic "vector potential" describes how to transport a quantum mechanical phase along a curve. Its representation as a vector field is somewhat arbitrary; only its curvature—the magnetic field—is a genuine vector field.

The mathematical formulation of a gauge field is as a connection in a vector bundle. A vector bundle is an assignment to each point of physical space of a complex vector space of some fixed dimension. The vector space associated with a point in physical space is called the fiber over the point. A section of a vector bundle is an assignment to each point of physical space of a vector in the corresponding fiber. A connection gives a method of differentiating sections along a curve in physical space. The vectors corresponding to different points on the curve are in different fibers. Thus one must first transport one of them so that the transported vector is in the same fiber as the other vector. Then the difference is well-defined, and one can subtract and finally take the limit to define the derivative.

In the case of a magnetic potential, each fiber has complex dimension one. The quantum mechanical phase at a point is a direction in the corresponding fiber. The group of unitary transformations of a one dimensional space is just a circle, an abelian group. This is the case of an *abelian gauge field*. The gauge fields that describe the interactions of elementary particles correspond to the case when the fibers have higher dimension. In this case the unitary transformations of a fiber are a nonabelian group, and so this is the case of a *nonabelian gauge field*. For nonabelian gauge fields the equations of motion are intrinsically nonlinear. So the program of using a linear field as a point of departure becomes dubious. Nevertheless, nonabelian gauge fields are central in elementary particle physics, and, as we shall see next, they may have advantages over other field theories.

Level 6. Asymptotic freedom.

The *renormalization group* is a dynamical system acting on the space of action functionals. It arises from integrations and changes of (distance or momentum) scale. The term group is misleading, since a dynamical system is the action of a one-parameter semigroup on a space. The semigroup can have a continuous parameter $t \geq 0$ or a discrete parameter $k = 0, 1, 2, 3, \ldots$. In the latter case, the action consists of the iterates T^k, $k = 0, 1, 2, 3, \ldots$ of a single transformation T. The renormalization group comes in both variants. The parameter has the interpretation of distance, measured on a logarithmic scale.

Here is an example of the discrete variant of the renormalization group. Let $a < b$ be distance scales, and let $\mu_{[a,b]}$ be the part of the field theory Gaussian measure that corresponds to integrating the Gaussian fluctuations on distances scales from a to b. Let ϕ_L be the scaling of the field ϕ with the property that

$$\int F(\phi_L)\, d\mu_{[a,b]}(\phi) = \int F(\phi)\, d\mu_{[La,Lb]}(\phi). \tag{24}$$

Fix $L > 1$. Let $S(\phi)$ be the non-quadratic part of the action functional. The renormalization group transformation T is

$$e^{-TS(\phi)} = \int e^{-S(\phi_L+\chi)}\, d\mu_{[a,La]}(\chi). \tag{25}$$

It follows easily that

$$e^{-T^k S(\phi)} = \int e^{-S(\phi_{L^k}+\chi)}\, d\mu_{[a,L^k a]}(\chi). \tag{26}$$

The effect is to integrate from a up to a scale $L^k a$ and then rescale. Think of a as a fixed small distance, an ultraviolet cutoff. Then $T^k S$ is the effective action at distance $L^k a$. One could fix the initial S and watch where the iterates lead as k increases. However the purpose of the renormalization group is to fix a final condition $T^k S$ and see what initial S could lead to this desired result. For this one needs a way of following the dynamics $S, TS, T^2S, T^3S, \ldots$ for arbitrary initial S.

Fix $b = L^k a$, and fix the parameters in $T^k S$ at reasonable values. If the corresponding parameters in S are small, then the original interaction at scale a was rather close to a linear (Gaussian) interaction, and so approximate calculations should be feasible. This situation is called *asymptotic freedom*. The other possibility is that the parameters in S must be quite large. This says that to get reasonable physical results one must start with an interaction that is far from linear. This looks like a hopeless situation. In particular, there may be no way to take a to zero and get a measure with no ultraviolet cutoff.

The condition of asymptotic freedom is a constraint on action functionals that purport to describe random fields or quantum fields with the required Euclidean or relativistic symmetry. The lectures of Gross in Part 2 indicate that, for dimension $n = 4$, the only fields with asymptotic freedom are nonabelian gauge fields. This is supporting evidence for their fundamental role. Rigorous work on renormalization group and asymptotic freedom is difficult, but there has been progress. For instance, there is a proof that the Gross-Neveu Fermi field model in dimension $n = 2$ is asymptotically free [4].

The renormalization group is also important in statistical mechanics, and it will be a central topic in the mathematical theory of probability for the next century or so. The hope is that it will give a way of getting a control on highly dependent systems, similar to the currently successful theory of independence and weak dependence.

Level 7. Broken symmetry.

Consider an action or energy functional

$$S_E(\phi) = \int \left(\frac{1}{2} |\nabla_x \phi|^2 + p(\phi) \right) d^n x. \tag{27}$$

Up to now we have taken for granted that this defines at most one measure

$$d\mu_E(\phi) = \frac{1}{Z} e^{-S_E(\phi)} \prod_x d\phi(x). \tag{28}$$

However that is not the case in general. Consider the case when the polynomial $p(\phi)$ is invariant under the reflection $\phi \mapsto -\phi$ and has two minima. An example is $p(\phi) = c\phi^2(\phi^2 - a^2)$, where $c > 0$ and $a > 0$. There are two functions that minimize the action, the constant function with value a and the constant function with value $-a$. This produces an ambiguity in the determination of the lowest energy state. There can be a similar ambiguity in the determination of the measure μ_E. This means that there are two measures, μ_E^+ and μ_E^-, each qualified to be reasonable interpretations of the formal expression for the measure. The one that occurs depends on boundary conditions that augment the specification of the problem. These two measures are sometimes called phases of the system, so the phenomenon is one of *multiple phases*. When the phases are related to a symmetry, such as the symmetry $\phi \mapsto -\phi$, then this is also called a *broken symmetry*.

Even when there is a symmetry of the action, it is not always the case that the measure displays broken symmetry. In dimension $n = 1$ the field is defined on the line, and occasional random fluctuations between the positive values and negative values of the field restore the symmetry. In dimension $n = 2$ broken symmetry can occur when the nonlinearities are strong enough. However the original symmetry must be discrete. This is the case in the example above, since the only non-trivial symmetry transformation is the reflection. In dimensions $n = 3$ and higher it is even possible to break continuous symmetry.

Multiple phases and broken symmetry play a profound role in statistical mechanics. They are also behind much of the reasoning about the use of quantum fields for the description of elementary particles. A highly symmetrical action can describe a quantum field with less symmetry, and this is needed to make theory match experiment.

Level 8. Supersymmetry.

There is a special class of theories in which there is an algebraic correspondence between the Bose and Fermi parts. This property is called *supersymmetry*. The usual kind of symmetry is generated by elements of a Lie algebra defined by relations involving the commutator bracket $[A, B] = AB - BA$. The new symmetries also involve relations involving the anticommutator bracket $\{A, B\} = AB + BA$. These two kinds of symmetries occur in the same algebraic system, hence the name supersymmetry.

The question is natural: Supersymmetries are supersymmetries of what? There is an interesting answer; they are symmetries of the superworld. The superworld is closely related to the usual world of algebra and geometry, but with this difference: all elements of an algebraic system are sums of even and odd (Bose and Fermi) elements. The characteristic property of the superworld is the *sign rule*: whenever the order in which two odd quantities appear is changed, a minus sign appears. Thus, for instance, the super Euclidean space with p even coordinates and q odd coordinates has coordinates $t_1, \ldots, t_p$ that commute and coordinates $\theta_1, \ldots, \theta_q$ that anti-commute. In other words, the odd coordinates form a Grassman algebra with the relation $\theta_j \theta_k = -\theta_k \theta_j$. In particular $\theta_j^2 = 0$. This explains why the anti-commuting coordinates might not be noticed; they are so small that their squares are zero.

For the simplest example of a supersymmetry, take $p = q = 1$. The superline has coordinates t and θ. The generator of ordinary translations is $\partial/\partial t$, and this is an even operator. Consider the odd operator

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t} \tag{29}$$

acting on functions of the coordinates (both even and odd). Since $\theta^2 = 0$ and $(\partial/\partial\theta)^2 = 0$, the anticommutator of D with itself is

$$\{D, D\} = 2D^2 = -2\frac{\partial}{\partial t}. \tag{30}$$

So the odd operator D is a supersymmetry generator that provides a square root of ordinary translation of the superline.

The supersymmetry relations themselves are an instance of the sign rule. Thus, for instance, the usual Lie bracket in a Lie algebra satisfies $[A, B] = -[B, A]$. In the superworld one has super Lie algebras. For two odd elements A, B we have an extra minus sign, so $[A, B] = [B, A]$. Thus we no longer need the notation $\{A, B\}$, we can just work in a super Lie algebra and keep track of whether A, B are even or odd. In the last example we would write $[D, D] = -2\partial/\partial t$, since D is odd.

The lectures of Deligne and Morgan in Part 1 give a systematic treatment of the properties of the superworld, including multilinear algebra, manifolds, and differential geometry. The following lectures of Deligne and Freed then cover the Lagrangian theory of classical fields, and finally they explain supersymmetry and give a large catalog of supersymmetric classical fields. In some cases a supersymmetric field may indeed be realized as a map whose domain is a super Euclidean (or super Minkowski) space. The authors are mathematicians; their treatment is sophisticated but careful, and they try to get the signs right (a nightmare in this subject).

We have already seen a hint of supersymmetry in the example of the one dimensional Bose field with action

$$S_E(\phi) = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + p(\phi) - E \right) dx = \frac{1}{2} \int_{-\infty}^{\infty} \left(\left(\frac{d\phi}{dx} \right)^2 + u(\phi)^2 + u'(\phi) \right) dx. \tag{31}$$

Why this strange representation of the $p(\phi) - E = (1/2)[u(\phi)^2 + u'(\phi)]$ interaction as the sum of a quadratic term and a derivative term? It turns out that this action is related to a supersymmetric action involving a potential function H related to u by $H' = u$. When this action is written in components, the $u(\phi)^2$ part is in a Bose term, and the $u'(\phi)$ part is in a Fermi term. The relation between the two terms is forced by the supersymmetry [5]. The idea of supersymmetry is to tease out the existence of such hidden relations in higher dimensions.

Witten is the master of applying quantum field theory to geometry. His lectures in Part 4 cover a wide variety of topics, but their culmination is a discussion of invariants of four dimensional manifolds. The mathematical relationships are uncovered by employing the entire apparatus of quantum field theory. In particular, the quantum fields that he uses involve nonabelian gauge fields and supersymmetry. Even if supersymmetry does not turn out to exist in Nature, it has earned a place in mathematics.

Level 9. Strings.

Gawędzki's lectures in Part 3 on conformal fields cover a special topic in quantum field theory that has various applications, including string theory. Conformal transformations are plentiful only in two dimensions, and for that reason *conformal field theory* starts with the study of quantum fields defined on two dimensional space or two dimensional space-time. Conformal transformations include dilations, and so the action should be invariant under changes of scale. The obvious possibility for a free field action with this invariance is

$$S_E(\phi) = \frac{1}{2} \int |\nabla_x \phi|^2 \, d^2x. \tag{32}$$

Since there are two derivatives and the integral is two dimensional, the action is invariant under changes of scale. Unfortunately, there is no such field defined on Euclidean space, since the covariance $(-\Delta)^{-1}$ does not exist in two dimensions. This is not as bad as it seems, since in the application to string theory (see below) the field is not defined on all of the Euclidean plane, but instead on a compact surface Σ. In fact, one of the main examples discussed by Gawędzki is the class of σ models. A σ *model* is a field defined on a manifold Σ with values in a Riemannian manifold M, often itself taken to be compact.

The fascination with conformal field theory is that the conformal symmetry is such a powerful constraint that it makes explicit computations feasible in many cases. The infinitesimal conformal transformations give rise to an infinite dimensional Lie algebra called the *Virasoro algebra.*

However conformal field theory is just one ingredient in *string theory.* String theory is a proposed replacement for quantum field theory. The relation between quantum field theory and string theory is the following. Quantum field theory is supposed to be a theory of point particles. A point particle moving in time is a parameterized curve in space-time. String theory is a theory of strings. A string moving in time is a parameterized surface in space-time.

In order to see the analogy, consider a point particle moving in a space (or space-time) manifold M. The path is a function $X(\tau)$, where τ ranges over a parameter interval of length L, and where $X(0) = x$. The values of this function are points in the space M of dimension n. Associate an action with such a function by the formula

$$S(X) = \frac{1}{2}\int_0^L \left|\frac{dX}{d\tau}\right|^2 \, d\tau. \tag{33}$$

Think of the length L of the interval as the time that the particle is allowed to diffuse. Then this action defines the usual Wiener measure of a diffusing particle. With the constraint $X(L) = y$ of fixed final point we have

$$\frac{1}{Z}\int e^{-S(X)}\delta(X(L)-y)\prod_\tau dX(\tau) = e^{\frac{1}{2}L\nabla^2}(x,y) = \frac{1}{(2\pi L)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{2L}}. \tag{34}$$

This is the usual Gaussian formula for the diffusion of a particle from x to y in time L. The integral of this Gaussian over all $L \geq 0$ gives the propagator

$$\frac{1}{2}\int_0^\infty e^{\frac{1}{2}L\nabla^2}(x,y)\, dL = (-\nabla^2)^{-1}(x,y) = \frac{1}{(n-2)a_n}\frac{1}{|x-y|^{n-2}} \tag{35}$$

from the point x to the point y. This is the usual propagator or covariance that enters into calculations in Euclidean field theory in the case $m = 0$ and $n > 2$.

The string theory version is similar. This time the functions are $X(\tau)$, where τ ranges over a two-dimensional surface Σ. The possible values are again points of the space (or space-time) M of dimension n, but the range traces out a surface. The action is that of a σ model:

$$S(X) = \frac{1}{2}\int_\Sigma |\nabla X|^2 \, d^2\tau. \tag{36}$$

However in addition there is an integral over the possible geometries and topologies of Σ. This takes the place of the simple integration over L in the example of the point particle.

There are refinements to this story, and D'Hoker describes a number of them in his section of Part 3. One is that the space-time M can be taken to have dimension considerably higher than four. The extra dimensions come from a compact manifold. Thus the action has to take into account the geometry and topology of space-time. Another refinement arises from the fact that the string theory described above is a theory of Bose particles. To get Fermi particles, one needs to have the concept of superstring. A superstring is a string in the superworld: the manifold M has both commuting and anti-commuting coordinates. Supersymmetry provides constraints on the theory, but it also forces nice properties. It brings in Bose and Fermi particles on an equal basis, and this in turn produces remarkable cancelations. One exciting consequence of string theory is that it provides a mechanism of unifying gravitation with the other known forces. D'Hoker's lectures conclude with supergravity formulated in superspace. There are several variants; in one the superspace M has 11 commuting coordinates and 32 non-commuting coordinates.

The incorporation of gravity makes string theory a candidate for an ultimate theory of physics. However, string theory as D'Hoker describes it is only a theory of a series expansion, where the order of the expansion is related to the topological complexity of Σ. Also, the experimental situation is bleak. Strings are predicted to be extraordinarily small; a probe of such short distances requires huge energy, many orders of magnitude beyond what current accelerators can provide.

Conclusion.

The lectures in this course aim to convey an understanding of quantum field theory. How well do they succeed? They contain an immense amount of valuable material on recent developments. The development of classical supersymmetry by Deligne and collaborators is careful and systematic. The chapters by Kazhdan are a terse but useful exposition of axiomatic quantum field theory. The course of Gawędzki on conformal field theory is a masterful treatment of a frontier subject. The lecture series by Witten, Gross, Fadeev, and D'Hoker present viewpoints of prominent theoretical physicists. Witten's lectures alone touch on such topics as renormalization, symmetry breaking, σ models, confinement, duality, solitons, gauge theories, quantum cohomology, and more. Other contributions may be helpful to specialists. There are hints at an overall unity, but it is elusive. Nevertheless, the book is a magnificent achievement. While it is difficult to imagine it as a first introduction to the subject, a reader with experience will find both general physics lore and information on specific models. Such a reader might also consult works (such as [1] and [2]) that give greater emphasis to the analytic and probabilistic side of the subject.

The search for the ultimate secret of Nature is not over. Quantum field theory and string theory answer old questions, only to open new ones. Dante encountered only nine levels of Hell. Field theorists may not be so lucky.

References

[1] Barry Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, New Jersey, 1974.

[2] James Glimm and Arthur Jaffe, Quantum Physics, A Functional Integral Point of View, second edition, Springer, New York, 1987.

[3] Roberto Fernández, Jürg Frölich, and Alan D. Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory, Springer, New York, 1992.

[4] Krzysztof Gawędzki and Antti Kupiainen, "Gross-Neveu model through convergent perturbation expansions," Commun. Math. Phys. 102 (1985), 1–30.

[5] Georg Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996.