A groupoid description for plane geometry

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Abstract

The transformational geometry approach to plane geometry involves the Euclidean group of isometries, a rather complicated three-dimensional non-commutative group with two connected components. In this approach constructions often involve choosing a particular element of this group to move a geometrical object from one location to another location. The group element may be described in various ways. A groupoid formulation give a particularly simple kind of description, one in which the move is directly described. This note explains the concept of groupoid and its application to this aspect of plane geometry.

0 Introduction

The teaching of plane geometry largely reflects the influence of Euclid, but Felix Klein introduced a new perspective in his Erlangen program of 1872. His emphasis was on the role of symmetry groups. The relevant group for plane geometry is generated by Euclidean transformations (isometries). Each component transformation is a translation, a rotation about some point, or a reflection in some line. The group approach appears in current school curricula under the name of transformational geometry. The book by Barker and Howe [1] contains a modern introduction to this subject. The book of Dodge [2] presents numerous examples of the use of Euclidean transformations to prove theorems in plane geometry.

The concept of groupoid generalizes the concept of group. (The article by Weinstein [6] gives an extensive discussion of groupoids and their role in describing symmetry. See also [4] for applications to geometry.) The present note argues that the groupoid concept can illuminate transformational geometry. A groupoid consists of a set of objects and a set of arrows. Each arrow describes a way to move from one object to another object. (The full definition of groupoid is given later in this introduction.) In the application to plane geometry the setting is a groupoid, a group, and a homomorphism from the groupoid onto the group. Each arrow specifies a group element, but various arrows may specify the same group element. The move given by an arrow is a more specific entity than the corresponding group element.

This is useful because the Euclidean group is a rather complicated object. It has a subgroup consisting of proper Euclidean transformations, those generated by translations and rotations alone. The remaining Euclidean transformation are called improper Euclidean transformations. Consider the following facts:

- Every proper Euclidean transformation is either a translation or a rotation about a point.
- Every improper Euclidean transformation is a glide-reflection.

(A glide-reflection is a reflection in some line together with a translation in the direction of the line.) The article [5] gives elegant proofs of these characterizations. They are less useful than one might think; in particular they require a construction to find the appropriate point or line. Consider, for instance, a task of rearranging furniture. If one wants to translate a chair from point P to point Q and then rotate it by a very small amount, this may be accomplished by a single rotation about a point C. But this point may be far away, perhaps across the street.

Another striking fact is the following:

- Every Euclidean transformation is a composition of reflections.

In particular, one can move the chair by reflecting it twice, a device worthy of Lewis Carroll. While this is fun to imagine, it is not the most natural way to proceed.

The groupoid notion gives a more practical way of describing Euclidean transformation. The basic definitions are elementary. A *groupoid* consists of a set of *objects* and a set of *arrows*. For each arrow there are two corresponding objects, the *source* of the arrow and the *target* of the arrow. We write $f : p \to q$ if arrow f has source p and target q. For each object p there is an *identity arrow* $1_p : p \to p$ with this object as source and target. Also, two arrows $f : p \to q$ and $g : q \to r$ determine an arrows $g \circ f : p \to r$ that is the *composition* of f and g. These operations are constrained by the following axioms.

Associative law Composition of arrows (wherever defined) satisfies the associative law.

Identities If $f : p \to q$, then $f \circ 1_p = 1_q \circ f = f$.

Inverses If $f : p \to q$, then there there is a unique *inverse arrow* $g : q \to p$ with $g \circ f = 1_p$ and $f \circ g = 1_q$.

The first two axioms are those of any category; the third axiom makes the category a groupoid.

A group is a groupoid with a single object $\bullet$. The arrows are the usual group elements. Thus there is a single identity $1_\bullet$, and every pair of elements may be composed.

The notion of groupoid homomorphism is defined in the obvious way. In particular, it makes sense to talk of a homomorphism of a groupoid into a group. Each object of the groupoid maps to $\bullet$, each 1_p maps to the group identity $1_\bullet$, and each $f : p \to q$ maps to a group element. The corresponding operations are preserved.

The remainder of this paper is organized around five examples of groupoids. For each example the groupoid is described by a typical (object, arrow) pair, and in each case there is a corresponding group of transformations. An arrow in the groupoid maps to a transformation in the group. The examples are listed below.

	groupoid	**group**
1.	(point, bound vector) $\widehat{\mathcal{T}}$	translation $\mathcal{T}$
2.	(fixed ray, proper move) $\widehat{\mathcal{O}}_P^+$	proper orthogonal $\mathcal{O}_P^+$
3.	(fixed ray, move) $\widehat{\mathcal{O}}_P$	orthogonal $\mathcal{O}_P$
4.	(ray, proper move) $\widehat{\mathcal{E}}^+$	proper Euclidean $\mathcal{E}^+$
5.	(ray, move) $\widehat{\mathcal{E}}$	Euclidean $\mathcal{E}$

In the groupoid examples a point is an element of the Euclidean space E. A fixed ray is a ray with vertex fixed at point P in E. A ray is a ray with arbitrary vertex in E. In the group examples a translation may be identified with a free vector. A proper orthogonal transformation is a rotation with fixed point P. An orthogonal transformation is a rotation or reflection with fixed point P. A proper Euclidean transformation is generated by translations and rotations, while a Euclidean transformation is generated by translations, rotations, and reflections. Euclidean transformations are quite complicated; the point of the following is that they have a simple groupoid description.

1 The (point, bound vector) groupoid

Many accounts of elementary vector analysis distinguish between bound vectors and free vectors. It is well known that the set of all free vectors forms a commutative group (in fact a vector space). The algebraic structure of the set of all bound vectors is more mysterious. It turns out that the natural structure is that of a groupoid. Futhermore, there is a groupoid homomorphism from the bound vectors to the free vectors.

In the following E denotes the Euclidean plane. It is not a vector space, but it is an affine space [3], and in addition it has a notion of Euclidean distance.

Groupoid 1. The basic example is the *(point, bound vector) groupoid* $\widehat{\mathcal{T}}$.

- An object is a point P in E.
- An arrow is an ordered pair of points PQ. This is also called a *bound vector*. The source is P and the target is Q.
- The composition of arrows is defined by $QR \circ PQ = PR$. Often this is written additively $PQ + QR = PR$.
- The identity arrow at P is PP.

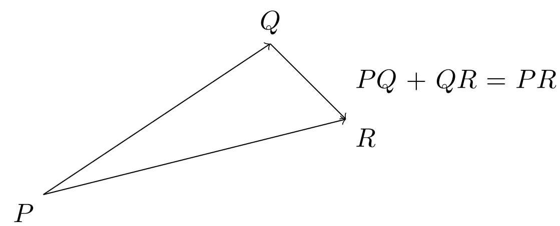

Figure 1: Objects are points; arrows are bound vectors.

- The inverse arrow to PQ is QP.

A bound vector is ordinarily pictured by an arrow leading from source point P to target point Q.

The corresponding group is a two-dimensional vector space $\mathcal{T}$. Each vector in this space defines a corresponding translation of E. This is a special kind of Euclidean transformation. An element of T is called also called a *free vector*. Thus a free vector $\mathbf{v}$ defines a function from the plane to itself that sends P to a new point, usually denoted by $P + \mathbf{v}$. In particular, a point P and a free vector $\mathbf{v}$ define a bound vector PQ, where $Q = P + \mathbf{v}$. A free vector $\mathbf{v}$ is ordinarily pictured in terms of many parallel arrows of the same length. These arrows may be thought of as ordered pairs that define the translation function. Addition of free vectors corresponds to composition of translation functions.

There is a homomorphism of the groupoid $\widehat{\mathcal{T}}$ of bound vectors to the group $\mathcal{T}$ of free vectors. The homomorphism sends the bound vector PQ to the corresponding free vector $Q - P$. This free vector $Q - P$ is the unique translation that sends P to Q. In particular $P + (Q - P) = Q$. $||$

Figure 1 illustrates the composition of two bound vectors to give a third bound vector. This is a trivial operation in itself, but it implies a corresponding composition of translations of Euclidean space.

Remark 1. The affine space operations on E come from corresponding operations on vectors. For instance, if P and Q are points, and $a + b = 1$, then the affine combination $aP + bQ$ is another point, defined by

$$aP + bQ = P + b(Q - P). \tag{1}$$

Many constructions with vectors and linear combinations have corresponding constructions with points and affine combinations. For instance, the line through P and Q consists of all $aP + bQ$ with $a + b = 1$. The ray $\overrightarrow{PQ}$ through P in the direction of Q consists of all $aP + bQ$ with $a + b = 1$ and $a \leq 1, b \geq 0$. $||$

2 The (fixed ray, proper move) groupoid

Many accounts of plane geometry distinguish between a directed angle (a geometrical figure) and the corresponding directed angle measure (a quantity of some sort). There is no problem defining addition of directed angle measures. There is also a notion of addition of directed angles, but this is only defined in special circumstances. This is groupoid addition, always present in elementary geometry, but seldom recognized as such.

A *ray* at point P in E is a set of points $P + t\mathbf{v}$, where $\mathbf{v}$ is a given non-zero free vector, and $t \geq 0$. Thus by definition a ray is fixed at a particular point. This is distinguished from a *ray direction* that consists of all vectors $t\mathbf{v}$, where $\mathbf{v}$ is a given non-zero vector, and $t \geq 0$. A ray may be thought of as a specified point P together with a specified ray direction r. It is convenient to denote such a ray by $P + r$. Often a ray is specified by giving the point P together with some other point R on the ray. A common notion for a ray described in this way is $\overrightarrow{PR}$. This ray is fixed at P.

A *directed angle* with vertex P consists of an ordered pair of rays $P+r, P+s$. Again this may be described with the notation $\overrightarrow{PR}, \overrightarrow{PS}$. In geometry a directed angle may be denoted by $\angle RPS$. The vertex is at P, and the point R, S occur in order.

The ordered pair of ray directions rs defines a directed angle measure θ. The directed angle measure θ is taken modulo $360°$. One possible notation for this angle is $\theta = s - r$.

Remark 2. Directed angle measures are typically expressed in degrees. There is an ambiguity in the definition of degree. This is because there are two possible orientations to the vector space. Typically these are called clockwise and counterclockwise. These are not absolute terms, since they depend on the perspective of an external observer. In any case, one can use degrees that correspond to one orientation or to the other orientation. Once this convention is established, then an expression like $90°$ has an unambiguous meaning. $||$

A directed angle $P + r, P + s$ defines a rotation $\mathcal{R}_P[rs]$ about the point P. This is the unique rotation about P that takes the first ray to the second ray. The rotation may also be written $\mathcal{R}_P[\theta]$, where θ is the directed angle measure. The relation between directed angle measure and rotation is defined by

$$\mathcal{R}_P[\theta_1 + \theta_2] = \mathcal{R}_P[\theta_1]\mathcal{R}_P[\theta_2] \tag{2}$$

Addition of directed angle measures (modulo $360°$) corresponds to composition of rotations.

Groupoid 2. Fix a point P in the Euclidean space E. The next example is the *(fixed ray, proper move) groupoid* $\widehat{\mathcal{O}}_P^+$. A ray fixed at P can move to another ray fixed at P, but only in one way, by a proper move.

- An object is a ray $P + r$.
- An arrow from $P + r$ to $P + s$ is the corresponding directed angle with vertex P. This arrow may be denoted by giving the ordered pair rs of ray directions.
- The composition of arrows is defined by $st \circ rs = rt$. This may also be written additively as $rs + st = rt$. This is the usual addition of directed angles.

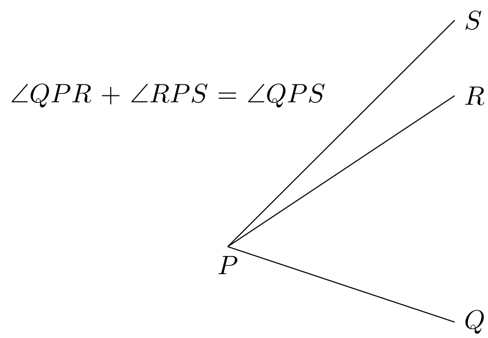

Figure 2: Objects are rays; arrows are directed angles.

- The identity arrow at $P + r$ is rr.
- The inverse arrow to rs is sr.

These operations may be described in another notation. If we denote the directed angles by $\angle RPS$ and $\angle SPT$, then the composition is $\angle RPS + \angle SPT = \angle RPT$. The identity directed angle is $\angle RPR$. The inverse to $\angle RPS$ is $\angle SPR$,

The corresponding group is the proper orthogonal group $\mathcal{O}_P^+$ of rotations about P, a one-dimensional commutative group. Each such rotation is characterized by specifying P and a directed angle measure θ. There is a corresponding rotation $\mathcal{R}_P[\theta]$, a special kind of Euclidean tranformation.

There is a homomorphism of the groupoid $\widehat{\mathcal{O}}_P^+$ to the corresponding rotation group $\mathcal{O}_P^+$. Each arrow rs determines the unique rotation $\mathcal{R}_P[rs]$ that takes ray $P + r$ to ray $P + s$. The corresponding angle may be written $\theta = s - r$. $||$

Figure 2 illustrates the composition of two directed angles; the result is a directed angle. This is a standard construction in geometry, naturally interpreted as groupoid composition. The corresponding group operation is addition of directed angle measures, or composition of the corresponding rotations.

3 The (fixed ray, move) groupoid

In elementary plane geometry there is also a notion of undirected angle. This notion has many uses. One application is to transformational geometry, where such an angle gives rise naturally to a reflection across the angle bisector line.

An (undirected) *angle* is a pair $P + r, P + s$ where the order of the two rays is not important. The corresponding angle measure $\pm\theta$ is only determined up to sign. A common convention is to measure the angle θ in unoriented degrees and to take $0^\circ \leq \theta \leq 180^\circ$.

A *line* L may described by a point P and a non-zero vector in the form $P + t\mathbf{v}$, where $\mathbf{v}$ is a non-zero free vector, and where t is real. It is convenient to define a *line direction* ℓ as a set of vectors of the form $t\mathbf{v}$, where $\mathbf{v}$ is a non-zero vector, and where t is real. In other words, it is a one-dimensional subspace of the space of free vectors. Then a line is given by a point P and a line direction ℓ. It is convenient to denote such a line by $L = P + \ell$.

A pair rs of ray directions defines a bisector line direction ℓ. This may be written $\ell = s \mid r$, and with this notation $s \mid r = r \mid s$. An angle given by rays $P+r$ and $P+s$ determines a line $L = P+\ell$ that is the angle bisector of the rays. This line in turn defines a reflection $\mathcal{H}[L] = \mathcal{H}_P[\ell]$ across the line. A reflection satisfies $\mathcal{H}_P[\ell]^2 = I$ and hence is its own inverse.

The relation between rotations and reflections is complicated, even when they all involve a single fixed point P. The formulas are

$$\begin{aligned} \mathcal{R}_P[\theta]\mathcal{R}_P[\phi] &= \mathcal{R}_P[\theta+\phi] \\ \mathcal{H}_P[\ell]\mathcal{R}_P[\theta] &= \mathcal{H}_P[\ell-\theta/2] \\ \mathcal{R}_P[\theta]\mathcal{H}_P[\ell] &= \mathcal{H}_P[\ell+\theta/2] \\ \mathcal{H}_P[m]\mathcal{H}_P[\ell] &= \mathcal{R}_P[2(m-\ell)]. \end{aligned} \tag{3}$$

An expression like $\ell \pm \theta/2$ is defined only up to a multiple of $180°$, but this is enough to define the rotated line. Similarly, $m - \ell$ denotes the directed angle measure between the lines of reflection. This is defined only up to multiples of $180°$, but the doubled directed angle measure $2(m-\ell)$ is well-defined. These are rather tricky relations. The groupoid description is much simpler.

Groupoid 3. Fix a point P in the Euclidean space E. The next example is the *(fixed ray, move) groupoid* $\widehat{\mathcal{O}}_P$. A ray fixed at P can move to another ray fixed at P in two possible ways.

- An object is a ray $P + r$, given by the fixed point P and the direction r.
- There are two arrows from $P+r$ to $P+s$, denoted rs and $r|s$. The second one may be read r "mirror" s.
- The composition of arrows is defined by

$$\begin{aligned} st \circ rs &= rt \\ s|t \circ rs &= r|t \\ st \circ r|s &= r|t \\ s|t \circ r|s &= rt. \end{aligned} \tag{4}$$

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- The identity arrow at $P + r$ is rr.
- The inverse arrow to rs is sr. The inverse arrow to $r|s$ is $s|r$.

The alternative notation works if we distinguish directed angle $\angle RPS$ from bisected angle $\angle|RPS$ (a directed angle with a mirror). Notice that $\angle RPR$ is the identity arrow at R, while $\angle|RPR$ is not the identity arrow at R.

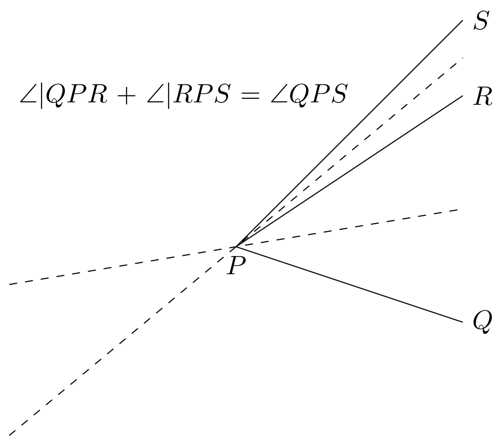

Figure 3: Objects are rays; arrows are directed angles or bisected angles.

The corresponding group is the orthogonal group $\mathcal{O}_P^+$, consisting of rotations and reflections leaving P invariant. This is a relatively complicated one-dimensional non-commutative group with two components. Each rotation is characterized by specifying P and a directed angle measure θ. There is a corresponding rotation $\mathcal{R}_P[\theta]$. Each reflection is characterized by specifying P and a line direction ℓ. There is a corresponding reflection $\mathcal{H}_P[\ell]$.

There is a homomorphism of the groupoid $\widehat{\mathcal{O}}_P$ to the corresponding orthogonal group $\mathcal{O}_P$. Each arrow rs determines the unique rotation $\mathcal{R}_P[rs]$ that takes ray $P+r$ to ray $P+s$. Each arrow $r|s$ determines the unique reflection $\mathcal{H}_P[r|s]$ that takes ray $P+r$ to ray $P+s$. This reflection is in the line $P+\ell$, where ℓ is the line direction bisecting the ray directions r, s. The reflections satisfy $\mathcal{H}_P[r|s] = \mathcal{H}_P[s|r]$, so they only depend on the undirected angle.

The formulas that express the homomorphism from the groupoid $\widehat{\mathcal{O}}_P$ to the group $\mathcal{O}_P$ are

$$\begin{aligned}
\mathcal{R}_P[st]\mathcal{R}_P[rs] &= \mathcal{R}_P[rt] \\
\mathcal{H}_P[s|t]\mathcal{R}_P[rs] &= \mathcal{H}_P[r|t] \\
\mathcal{R}_P[st]\mathcal{H}_P[r|s] &= \mathcal{H}_P[r|t] \\
\mathcal{H}_P[s|t]\mathcal{H}_P[r|s] &= \mathcal{R}_P[rt]. \qquad (5)
\end{aligned}$$

These give a simple and uniform description of the group $\mathcal{O}_P$. $||$

Figure 3 illustrates the composition of two bisected angles; the result is a directed angle. This captures the remarkable fact that the composition of two mirror reflections in line directions related by directed angle measure θ is a rotation by directed angle measure 2θ.

4 The (ray, proper move) groupoid

Groupoid 4. A widely useful example is the *(ray, proper move) groupoid* $\widehat{\mathcal{E}}^{+}$. A ray can move to another ray, but in only way, by a proper move.

- An object is a ray $P + r$ determined by point P and direction r.
- There is just one arrow from $P + r$ to $Q + s$. This is $PQrs$.
- The composition of arrows is defined by $QRst \circ PQrs = PRrt$.
- The identity arrow at $P + r$ is $PPrr$.
- The inverse arrow to $PQrs$ is $QPsr$.

The corresponding group is the group $\mathcal{E}^{+}$ of proper Euclidean transformation. This is a rather complicated three-dimensional non-commutative group, but the groupoid description does a lot to tame this complexity.

There is a homomorphism of the groupoid $\widehat{\mathcal{E}}^{+}$ to the corresponding proper Euclidean group $\mathcal{E}^{+}$. Each arrow $PQrs$ determines the unique proper Euclidean transformation that takes ray $P+r$ to ray $Q+s$. A simple groupoid computation shows how this works. There is an identity

$$PQrs = PQss \circ PPrs = QQrs \circ PQrr. \tag{6}$$

The corresponding relation for the transformations is the map $X \mapsto \mathcal{A}(X)$ from the plane E to itself given by

$$\mathcal{A}(X) = \mathcal{R}_P[rs](X) + Q - P = \mathcal{R}_Q[rs](X + Q - P). \tag{7}$$

Rotation about P followed by translation is the same as translation followed by rotation about Q. $||$

5 The (ray, move) groupoid

Groupoid 5. The most general example is the *(ray, move) groupoid* $\widehat{\mathcal{E}}$. A ray can move to another ray in two possible ways.

- An object is a ray $P + r$ determined by point P and direction r.
- There are always two arrows from $P + r$ to $Q + s$. These are $PQrs$ and $PQr|s$.
- The composition of arrows is defined by

$$\begin{aligned} QRst \circ PQrs &= PRrt \\ QRs|t \circ PQrs &= PRr|t \\ QRst \circ PQr|s &= PRr|t \\ QRs|t \circ PQr|s &= PRrt. \end{aligned} \tag{8}$$

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- The identity arrow at $P + r$ is $PPrr$.
- The inverse arrow to $PQrs$ is $QPsr$. The inverse arrow to $PQr|s$ is $QPs|r$.

The corresponding group is the Euclidean group $\mathcal{E}$, which is a three-dimensional non-commutative group with two components. This group is the main subject of planar transformational geometry. Again it may be tamed by the groupoid description.

There is a homomorphism of the groupoid $\widehat{\mathcal{E}}$ to the corresponding Euclidean group $\mathcal{E}$. Each arrow $PQrs$ determines the unique proper Euclidean transformation that takes ray $P + r$ to ray $Q + s$. Each arrow $PQr|s$ determines the unique improper Euclidean transformation that takes ray $P + r$ to ray $P + s$.

The explicit forms of these transformations arise from simple groupoid computations. Thus

$$PQr|s = PQss \circ PPr|s = QQr|s \circ PQrr. \tag{9}$$

The representation by transformations is the map $X \mapsto \mathcal{A}(X)$ from the plane E to itself given by

$$\mathcal{A}(X) = \mathcal{H}_P[r|s](X) + Q - P = \mathcal{H}_Q[r|s](X + Q - P). \tag{10}$$

This says that you can reflect in the rays and then translate, or you can translate and then reflect in the rays.

Another useful and natural relation is

$$PQr|s = PQrs \circ PPr|r = QQs|s \circ PQrs. \tag{11}$$

The representation by transformations is the map $X \mapsto \mathcal{A}(X)$ from the plane E to itself given by

$$\mathcal{A}(X) = \mathcal{R}_P[rs]\mathcal{H}_P[r|r](X) + Q - P = \mathcal{H}_Q[ss]\mathcal{R}_Q[rs](X + Q - P). \tag{12}$$

This says that you can reflect in the source ray and then apply a proper Euclidean transformation, or you can apply a proper Euclidean transformation and then reflect in the target ray. $||$

6 Conclusion

The Euclidean transformations of plane geometry may be described in various ways. A common problem is to find a description of a Euclidean transformation that takes a geometric figure based at one point to a geometric figure based at another point. This often reduces to finding a transformation that takes a ray at one point to a ray at another point. There are two such transformation. The simplest proper Euclidean transformation involves translation and rotation about a point. There is more than one possible choice of point. However the rotation must be by the directed angle measure θ defined by the two rays. The simplest improper Euclidean transformation involves translation and reflection in a line. There is more than one possible choice of line. However the line must be aligned with a line direction ℓ that bisects the two ray directions.

This has a precise mathematical formulation in the groupoid language. For each ordered pair of rays $P + r$ and $Q + s$ there are exactly two arrows in the groupoid, one proper, and the other improper. Each groupoid arrow is represented by the unique proper or improper Euclidean transformation that takes the first ray to the second ray. The groupoid structure is the following. In every case an arrow from $P + r$ to $Q + s$ followed by an arrow from $Q + s$ to $R + t$ is an arrow from $P + r$ to $R + t$. The proper-improper structure is given by the table

$$\begin{aligned} \text{proper} \circ \text{proper} &= \text{proper} \\ \text{improper} \circ \text{proper} &= \text{improper} \\ \text{proper} \circ \text{improper} &= \text{improper} \\ \text{improper} \circ \text{improper} &= \text{proper}. \end{aligned} \tag{13}$$

(This is just the algebra of multiplying $+1$ and -1.) The result is a remarkably simple formulation of transformational geometry.

References

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