Gaussian processes and Feynman diagrams

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April 25, 2013

1 Introduction

These talks are about expectations of non-linear functions of Gaussian random variables. The first talk presents the famous Feynman diagram expansion, which is straightforward but for which it is difficult to obtain convergence results. The second talk is about another approach that leads to convergent expressions, at least in a small parameter regime.

The story begins with an integral of the form

$$Z = \langle e^{S(\phi)} \rangle. \tag{1}$$

Here ϕ refers to a family of mean-zero Gaussian random variables ϕ_x for x in a set $\mathcal{X}$. The brackets denote expectation with respect to the Gaussian measure. The Gaussian variables have covariance $\langle \phi_x \phi_y \rangle = C_{xy}$. The function $S(\phi)$ is expressed as a power series in these random variables; for the moment we take $S(0) = 0$. Often it is just a polynomial, for instance one useful test case is

$$S(\phi) = \sum_x s_x \phi_x + \sum_x \sum_{n=2}^{m} s_n \phi_x^n \tag{2}$$

with leading coefficient $s_m < 0$. This is the sum of a general first order term with higher order terms whose coefficients do not depend on x.

The eventual goal is to get an expression for a quantity F such that

$$Z = \langle e^{S(\phi)} \rangle = e^F. \tag{3}$$

One motivation for this problem that this gives an approach to defining a new non-Gaussian probability model. The idea is simple and is explained in the final part of this talk. For the moment accept that computation of F is an important goal.

There is an analogous problem in quantum field theory. This problem is much more singular, for many reasons. In the present context the important quantities are positive, so we are in the familiar context of probability theory. In quantum field theory the corresponding quantities are complex numbers. Worse, the role of $\mathcal{X}$ is played by space-time, which is a continuum. This leads to the notorious divergences due to short-distance behavior.

It is possible to do the probability problem in a context where $\mathcal{X}$ is Euclidean space. There are analytic continuation results that show that this is relevant to the study of quantum field theory. But even in the probability context there are serious issues with short distance behavior.

2 Power series in several variables

In the following we will often have the following setup. There is a given set $\mathcal{X}$. There may also be a mapping $a : U_n \to \mathcal{X}$. Here U_n is a set with n elements. For instance, one possibility for U_3 is $\{4, 5, 7\}$. Then there would be corresponding elements $a(4), a(5), a(7)$ of $\mathcal{X}$, which might or might not be distinct. For each such mapping there is a corresponding function N on $\mathcal{X}$ that counts the number of i such that $a(i) = x$.

The terminology for U_n and $\mathcal{X}$ varies from situation to situation.

- In tensor terminology U_n is an index set, and $\mathcal{X}$ the coordinate set. A mapping a is a listing of some of the coordinates. The corresponding N is a multi-index.
- In combinatorics terminology U_n is the label set, and $\mathcal{X}$ the color set. A mapping a is a coloring of the set U_n using color palette $\mathcal{X}$. The corresponding N is the inventory of colors.
- In physics U_n is a set of n particles, and $\mathcal{X}$ is the set of locations. A mapping a is a particle configuration in the discrete space $\mathcal{X}$. The corresponding N is the occupation number function.

Fix a set $\mathcal{X}$. This set will index coordinates. Let z_x for $x \in \mathcal{X}$ be corresponding variables. An *exponential generating function* is a (formal) power series

$$F(z) = \sum_N \frac{1}{N!} f(N) z^N. \tag{4}$$

Here N is a multi-index defined on $\mathcal{X}$, that is, $N : \mathcal{X} \to \{0, 1, 2, 3, \ldots\}$. The factorial $N!$ is defined by $N! = \prod_x N(x)!$. The power $z^N = \prod_x z_x^{N(x)}$. There is nothing special here: this is just a general power series expansion in many variables. The coefficient $f(N)$ is the derivative $D^N F$ evaluated at zero. The expression above is called the *multi-index form* of the series.

The same series may be written in another notation. In this notation there is an index set U_n for each n. This is a set with n elements. If we have $a : U_n \to \mathcal{X}$, then we have a corresponding multi-index $[\#a]$ given by

$$[\#a](x) = \#\{i \mid a(i) = x\}. \tag{5}$$

The number of colored sets with given multi-index is given by the multinomial coefficient

$$\#\{a \mid [\#a] = N\} = \frac{n!}{N!}. \tag{6}$$

We write

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} f(a) \prod_{i \in U_n} z_{a(i)}. \tag{7}$$

Here $f(a) = f([\#a])$. This is called the *symmetric tensor form.* It is identical to the multi-index form. One can make it look more familiar by writing it as

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a_1,\ldots,a_n} f_{a_1,\ldots,a_n} z_{a_1} \cdots z_{a_n}. \tag{8}$$

In many situations the symmetric tensor form is the more natural setting for combinatorics.

3 The exponential of a power series

Here is a fundamental relation in combinatorics. Say that $G(0) = 1$ and $C(0) = 0$ and

$$G(z) = \exp(C(z)). \tag{9}$$

This is the exponential of a function. The combinatorial issue concerns the relation between the corresponding coefficients in the series expansion.

Write $\mathrm{Part}[U]$ for the set of set partitions of U (into non-empty non-overlapping subsets). Then

$$g(a) = \sum_{\Gamma \in \mathrm{Part}[U_n]} \prod_{V \in \Gamma} c(a_V), \tag{10}$$

where a_V is the restriction of a to V. In this representation $C(z)$ is called the connected function, and the coefficients $c(a)$ are the cluster coefficients. This relation is called the combinatorial exponential. It is a relation between the coefficients. It is vital to distinguish the exponential from the combinatorial exponential.

The combinatorial exponential has a multi-index form, that gives more detailed information, but at the price of complication. The statement is that

$$g(N) = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{M_1+\cdots+M_p=N} \frac{N!}{M_1! \cdots M_p!} c(M_1) \cdots c(M_p). \tag{11}$$

The sum is over sequences of multi-indices. Each such sequence produces a higher multi-index k that counts how many terms in the sequence equal a given multi-index M. The number of sequences with given k satisfying $\sum_M k(M) = p$ is the multinomial coefficient $p!/\prod_M k(M)!$. So we can write this as a sum over such k in the form

$$g(N) = \sum_{\sum_M k(M)M=N} \frac{N!}{\prod_M k(M)! \prod_M (M!)^{k(M)}} \prod_M c(M)^{k(M)}. \tag{12}$$

As an example of the exponential construction, consider the exponential of a quadratic

$$C(z) = \frac{1}{2}\sum_{a_1}\sum_{a_2} c(a) z_{a_1} z_{a_2} = \sum_{|N|=2} \frac{1}{N!} c(N) z^N. \tag{13}$$

Then

$$g(a) = \sum_{\sigma \in \mathrm{Match}[U_n]} \prod_{W \in \sigma} c(W). \tag{14}$$

This is a sum over partitions into two-point blocks, that is, a sum over perfect matchings.

The multi-index form gives a sum over graphs with vertex set $\mathcal{X}$ having multiple edges and loops. Such a graph may be thought of as a multiplicity function G defined on simple edges $\ell = \{x, y\}$, where $x = y$ is allowed for a loop. The multiplicity of an edge ℓ is $G(\ell)$. Each edge has two end-points. A graph G has degree function N if the number of end points of edges at each vertex x is $N(x)$. Thus

$$g(N) = \sum_{\mathrm{degree}(G)=N} \frac{N!}{\prod_\ell G(\ell)! 2^{\mathrm{cycle}(G)}} \prod_\ell c(\ell)^{G(\ell)}. \tag{15}$$

Also

$$G(z) = \sum_G \frac{1}{\prod_\ell G(\ell)! 2^{\mathrm{cycle}(G)}} \prod_\ell c(\ell)^{G(\ell)} z^{\mathrm{degree}(G)}. \tag{16}$$

This is a sum over all graphs.

It is possible to write $G(z)$ as a sum involving simple graphs with loops (without multiplicities). However this requires a different expansion parameter. The result is

$$G(z) = \sum_G \prod_{\{x,y\} \in G} (e^{\bar{c}(x,y) z_x z_y} - 1), \tag{17}$$

where $\bar{c}(x, y) = c(x, y)$ for $x \neq y$ and $\bar{c}(x, x) = \frac{1}{2} c(x, x)$. If we expand the exponential we get the contributions of the multiple edges.

4 Moments and cumulants

In one famous interpretation of this formula the $g(a)$ are the moments of a probability distribution, and the $c(a)$ are the corresponding cumulants. In other words, we have random variables ϕ_x with

$$G(z) = \langle \exp(\sum_x z_x \phi_x) \rangle = \exp(C(z)). \tag{18}$$

Then the moments are

$$g(a) = \langle \prod_i \phi_{a_i} \rangle \tag{19}$$

and the cumulants are the corresponding $c(a)$.

The mean-zero Gaussian case is when all cumulants are zero except for the case $n = 2$. In that case for $a : \{1, 2\} \to \mathcal{X}$ the cumulant $c(a) = C_{a_1\,a_2}$, where C is the covariance matrix. So we have the Gaussian relation

$$\langle \exp(\sum_p z_p \phi_p) \rangle = \exp(\frac{1}{2} \sum_{a_1\,a_2} C_{a_1\,a_2} z_{a_1} z_{a_2}). \tag{20}$$

In particular, for the moments corresponding to $a : U_n \to \mathcal{X}$ we have

$$\langle \phi_{a_1} \cdots \phi_{a_n} \rangle = \sum_{\sigma \in \mathrm{Match}[U_n]} \prod_{W \in \sigma} c(a_W). \tag{21}$$

Here $\mathrm{Match}[U_n]$ consists of all perfect matchings of U_n, that is, of set partitions of U_n into blocks of size 2. For each block $W = \{i, j\}$ there is a corresponding covariance $c(a_W) = C_{a_i\,a_j}$ and the contribution of a particular matching partition is the product of these covariances.

We could also write this in the multi-index form as

$$\langle \phi^N \rangle = \sum_{\mathrm{degree}(G)=N} \frac{N!}{\prod_\ell G(\ell)! 2^{\mathrm{cycle}(G)}} \prod_\ell c(\ell)^{G(\ell)}. \tag{22}$$

There is a much more general formula that gives (at least in principle) the expectation of an arbitrary random variable with respect to the Gaussian probability measure. Let

$$\Delta = \sum_x \sum_y \Delta_{xy} = \sum_x \sum_y C_{xy} \frac{\partial}{\partial \phi_x} \frac{\partial}{\partial \phi_y}. \tag{23}$$

The formula is

$$\langle f(\phi) \rangle = (e^{\frac{1}{2}\Delta} f)(0). \tag{24}$$

The crudest way to use this formula is to expand the exponential. The partial derivatives all commute, and so they may be treated like variables. Then one-half the Laplace operator is of the same form as the exponential generating function for two point subsets with weights C_{xy} given by the corresponding colors. It follows that $\exp(\frac{1}{2}\Delta)$ is given by the usual exponential formula that corresponds to partitions of an m element set into two-point subsets:

$$\langle f(\phi) \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{a : U_m \to \Lambda} \sum_{\sigma \in \mathrm{Match}[U_m]} \prod_{\{i,j\} \in \sigma} C_{a_i\,a_j} (\prod_{j \in U_m} \frac{\partial}{\partial \phi_{a_j}} f)(0). \tag{25}$$

Alternatively, we can write

$$\langle f(\phi) \rangle = \sum_G \frac{1}{\prod_\ell G(\ell)! 2^{\mathrm{cycle}(G)}} \prod_\ell c(\ell)^{G(\ell)} (D^{\mathrm{degree}(G)} f)(0). \tag{26}$$

This is a sum over graphs where edges have weights given by covariances, and the vertices have weights given by partial derivatives of the function. The partial derivatives associated with a vertex are determined by the corresponding end points of edges at that vertex.

As a trivial illustration of this expansion, take the case when $f(\phi) = \phi^M$ is a monomial. Then $D^N\phi^M$ evaluated at 0 is zero unless $N = M$, and in that case $D^M\phi^M = M!$. So we conclude that $\langle\phi^M\rangle = g(M)$ as before.

5 Gaussian process Feynman diagrams

In Euclidean field theory there is a central problem. Start with Gaussian random variables ϕ_p. Consider the *action*

$$S(\phi) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{a:U_n\to\mathcal{X}} s(a) \prod_{i\in U_n} \phi_{a_i}. \tag{27}$$

Sometimes this is just a polynomial of reasonably low degree, but greater than 2. The problem is to calculate the expectation

$$Z = \langle\exp(S(\phi))\rangle \tag{28}$$

and various related quantities. A standard calculation gives

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n\to\mathcal{X}} \sum_{\Delta\in\mathrm{Part}[U_n]} \prod_{V\in\Delta} s(a_V)\langle \prod_{i\in U_n} \phi_{a_i}\rangle \tag{29}$$

However since these are Gaussian random variables this is

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n\to\mathcal{X}} \sum_{\Delta\in\mathrm{Part}[U_n]} \sum_{\sigma\in\mathrm{Match}[U_n]} \prod_{V\in\Delta} s(a_V) \prod_{W\in\sigma} c(a_W). \tag{30}$$

A Feynman diagram considers of a set U_n, a set partition Δ of U_n, and a perfect matching σ of U_n. A block V in Δ is called a vertex, while a two-element set W in σ is called a line. The corresponding covariance C_{a_i,a_j} propagates the influence from a_i to a_j. A vertex factor $s(a_V)$ describes the non-linear interaction taking place at all the a_k for $k \in V$.

Each Feynman diagram defines a graph (possibly with multiple edges and loops). The vertices of the graph are the blocks in Δ. The edges of the graph are the two-element sets $W = \{i, j\}$ in the perfect matching. An edge $\{i, j\}$ connects the blocks of Δ to which i and j belong. The topology of these graphs depends on the details of the interaction. For instance, if $S(\phi)$ is a polynomial of degree m, then every vertex will have degree at most m.

6 Connected diagrams

A Feynman diagram $D = (\Delta, \sigma)$ always has a decomposition into connected components. This consists of a maximal partition Γ of U_n such each vertex in

Δ and each line in σ is a subset of some block V in Γ. Thus we get connected diagrams on each block V of Γ.

We want to write $Z = e^F$. Write a diagram as $D = (\Delta, \sigma)$ with

$$\mathrm{wt}(a, D) = \prod_{V \in \Delta} s(a_V) \prod_{W \in \sigma} c(a_W). \tag{31}$$

Then

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} z(a), \tag{32}$$

where

$$z(a) = \sum_D \mathrm{wt}(a, D). \tag{33}$$

It follows that

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{X}} f(a), \tag{34}$$

where

$$f(a) = \sum_{D_c} \mathrm{wt}(a, D_c). \tag{35}$$

This is the sum over connected diagrams. This combinatorial device gives a way of writing Z in the form of an exponential $Z = e^F$. This is a standard technique in both combinatorics and physics: write the coefficients of Z as a sum of objects built over a set, discover the coefficients of F as the corresponding sum of connected objects.

To see why this works, one can calculate the combinatorial exponential

$$\sum_\Gamma \prod_{V \in \Gamma} f(a_V) = \sum_\Gamma \prod_{V \in \Gamma} \sum_{D_c} \mathrm{wt}(a_V, D_c) = \sum_\Gamma \sum_\chi \prod_{V \in \Gamma} \mathrm{wt}(a_V, \chi(V)) \tag{36}$$

where χ assigns to each block a corresponding connected diagram. But each diagram D corresponds to a partition Γ and a connected diagram $\chi(V)$ on each block in the partition. Furthermore, the weight of the diagram is the product of the weights of the connected diagrams. So this is just

$$\sum_D \mathrm{wt}(a, D) = z(a). \tag{37}$$

Given the expansion for F, one can ask about whether it (or various related series) have any reasonable convergence properties. Perhaps a good test case is to take

$$S(\phi) = \sum_x s_x \phi_x + \sum_x s_4 \phi_x^4. \tag{38}$$

The reason for the fourth power is that this is the first even power greater than two. This example is the case where all vertices of the diagrams are of degree one or four. The sign of the leading coefficient s_4 is crucial. If it is negative, then $S(\phi)$ is bounded above, and $\exp(-S(\phi))$ is bounded. If it is positive, then the whole enterprise looks questionable. This is at least one reason why one might not expect analyticity. The actual series expansion has lots of terms, and it appears that the terms must have both signs. An attempt to extract rigorous information from the diagram expansion must exploit cancelation.

7 The probability model

Here is the explanation of the probability model mentioned in the introduction. Write

$$S(\phi) = \sum_x s_x \phi_x + S_{\geq 2}(\phi). \tag{39}$$

Then

$$Z(s) = \langle e^{S(\phi)} \rangle = \langle e^{\sum_x s_x \phi_x} e^{S_{\geq 2}(\phi)} \rangle = e^{F(s)}. \tag{40}$$

This looks like the moment generating function for a non-Gaussian measure whose density with respect to the Gaussian measure is proportional to $e^{S_{\geq 2}(\phi)}$. There is a serious problem: this is not properly normalized. However we can write

$$\frac{Z(s)}{Z(0)} = e^{F(s)-F(0)}. \tag{41}$$

The left hand side is a moment generating function, but its definition involves an unpleasant division. The $F(s) - F(0)$ on the right hand side is a cumulant generating function that only involves an innocuous subtraction.

We have seen that $Z(s)$ is the sum of contributions from all Feynman diagrams. Then $F(s)$ is the contribution from connected Feynman diagrams. Similarly, $Z(0)$ is the sum of all contributions from Feynman diagrams without one point blocks, and $F(0)$ is the sum of all contributions from connected Feynman diagrams without one-point blocks. The cumulant generating function $F(s) - F(0)$ is the sum of connected Feynman diagrams each one of which has at least one one-point block.

Usually the convergence results are presented not for F, but for the cumulants. For $N \neq 0$ these are

$$(D^N F)(0) = (D^N \log Z)(0). \tag{42}$$

The advantage is that these quantities are pinned at particular points. For example, the first logarithmic derivative gives the expectation

$$(\frac{\partial}{\partial s_x} F)(0) = \frac{1}{Z(0)} \langle \phi_x e^{S_{\geq 2}(\phi)} \rangle. \tag{43}$$

The next derivative gives the covariance

$$(\frac{\partial^2}{\partial s_x \partial s_y} F)(0) = \frac{1}{Z(0)} \langle \phi_x \phi_y e^{S_{\geq 2}(\phi)} \rangle - \frac{1}{Z(0)} \langle \phi_x e^{S_{\geq 2}(\phi)} \rangle \frac{1}{Z(0)} \langle \phi_y e^{S_{\geq 2}(\phi)} \rangle. \tag{44}$$