Propagation of singularities for the wave equation

Let $u(t,x)$ be the solution of the initial value problem for the wave equation in $\mathbf{R}^n$:

$$u_{tt} = \Delta u, \quad u(0,x) = 0, \quad u_t(0,x) = f(x). \tag{1}$$

At this point, I assume that $f(x) \in L^2(\mathbf{R}^n)$ and that the function $f(x)$ has compact support. In the end of these notes, I discuss the general case $f \in \mathcal{D}'(\mathbf{R}^n)$. To solve the problem (1), one makes the Fourier transform $u(t,x) \mapsto \hat{u}(t,\xi)$, $f(x) \mapsto \hat{f}(\xi)$. Then

$$\hat{u}_{tt} + |\xi|^2 \hat{u} = 0, \quad \hat{u}(0,\xi) = 0, \quad \hat{u}_t(0,\xi) = \hat{f}(\xi).$$

The last problem can be easily solved:

$$\hat{u}(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|} \hat{f}(\xi). \tag{2}$$

Notice that formula (2) makes perfect sense when f is a distribution; then $\hat{f}$ is a distribution, and $\hat{u}(t)$ is a distribution for any fixed value of t. Our goal is to localize the wave front set of $u(t)$ in terms of the wave front set of f. By $u(t)$ I denote the function $u(t,\cdot)$.

The formula (2) implies immediately that $\hat{u}(t,\xi)$ is rapidly decaying (by "rapidly decaying" I always mean "decaying faster than any negative power of $1+|\xi|$") exactly in the directions where $\hat{f}(\xi)$ is rapidly decaying. Therefore

$$\Sigma(u(t)) = \Sigma(f). \tag{3}$$

In particular, the solution is a smooth function if the initial condition is smooth. We take the inverse Fourier transform of $\hat{u}(t,\xi)$ to get

$$u(t,x) = (2\pi)^{-n} \int \frac{\sin(t|\xi|)}{|\xi|} e^{ix\xi} \hat{f}(\xi) d\xi. \tag{4}$$

It is convenient to express the sin function as a combination of exponential functions and to break the integral (4) into the sum of two integrals. However, if one does this then each integral will be divergent in a neighborhood of $\xi = 0$. To avoid this problem, I introduce a cut-off function $\chi(\xi)$. This is a smooth function that equals 1 when $|\xi| \geq 1$, and it vanishes when $|\xi| \leq 1/2$. Then

$$u(t,x) = \frac{(2\pi)^{-n}}{2i}(u_+(t,x) - u_-(t,x)) + (2\pi)^{-n} \int (1-\chi(\xi)) \frac{\sin(t|\xi|)}{|\xi|} e^{ix\xi} \hat{f}(\xi) d\xi \tag{5}$$

where

$$u_\pm(t,x) = \int \frac{\chi(\xi)}{|\xi|} e^{i[(x-y)\xi \pm t|\xi|]} \hat{f}(\xi) d\xi. \tag{6}$$

The function $1-\chi(\xi)$ vanishes outside the unit ball, so the last term in (5) represents a smooth function. Therefore, $WF(u(t)) \subset WF(u_+(t)) \cup WF(u_-(t))$. Notice that $u_-(t) = u_+(-t)$, so it is sufficient to study $WF(u_+(t))$; then $WF(u_-(t)) = WF(u_+(-t))$.

First, we prove the following lemma.

Lemma 1. *Let* $\Sigma_1(f) = \Sigma(f) \cap \{\xi : |\xi| = 1\}$. *Then*

$$\text{sing supp}(u_+(t)) \subset \text{supp}(f) - t\Sigma_1(f). \tag{7}$$

For two sets $A, B \subset \mathbf{R}^n$, by $A - B$ we denote the set of all differences $x - \xi$ where $x \in A$ and $\xi \in B$.

Proof. Let x_0 be a point that does not belong to $\text{supp}(f) - t\Sigma_1(f)$. We have to show that $x_0 \notin \text{sing supp}(u_+(t))$. Choose a neighborhood U of the point x_0 and a conic neighborhood Γ of $\Sigma(f)$ such that

$$U \cap (\text{supp}(f) - t\Gamma_1) = \emptyset; \tag{8}$$

here $\Gamma_1 = \Gamma \cap \{\xi : |\xi| = 1\}$. We introduce a cut-off function $\psi(\xi)$ that satisfies the following properties:

(i) $\psi(\xi) \in C^\infty(\mathbf{R}^n \setminus \{0\})$;

(ii) $\psi(\xi)$ is homogeneous of degree 0, i.e. $\psi(\tau\xi) = \psi(\xi)$ for every $\tau > 0$;

(iii) $\psi(\xi) = 1$ when ξ belongs to a neighborhood of $\Sigma(f)$;

(iv) $\psi(\xi) = 0$ when $\xi \notin \Gamma$.

We break the integral that represents $u_+(t,x)$ (see (6)) into the sum $u_+^1(t,x) + u_+^2(t,x)$ where

$$u_+^1(t,x) = \int \frac{\psi(\xi)\chi(\xi)}{|\xi|} e^{i(x\xi + t|\xi|)} \hat{f}(\xi) d\xi \tag{9}$$

and

$$u_+^2(t,x) = \int \frac{(1-\psi(\xi))\chi(\xi)}{|\xi|} e^{i(x\xi + t|\xi|)} \hat{f}(\xi) d\xi.$$

The Fourier transform $\hat{f}(\xi)$ of the function f is a rapidly decaying function on the support of $1-\psi(\xi)$, so $u_+^2(t,x)$ is a smooth function, and the singular support of $u_+(t,x)$ is the same as of $u_+^1(t,x)$. Now we use the Fourier transform formula for $\hat{f}(\xi)$ to rewrite (9) in the form

$$u_+^1(t,x) = \int\int \frac{\psi(\xi)\chi(\xi)}{|\xi|} e^{i[(x-y)\xi + t|\xi|]} f(y) dy d\xi. \tag{10}$$

We will do a number of partial integrations in (10). Let $D_j = D_{\xi_j} = (1/i)\partial/\partial\xi_j$. One has

$$D_j e^{[(x-y)\xi + t|\xi|]} = \left(x - y + t\frac{\xi}{|\xi|}\right)_j e^{[(x-y)\xi + t|\xi|]}. \tag{11}$$

Introduce a first order differential operator

$$L = \sum_{j=1}^n \frac{(x - y + t(\xi/|\xi|))_j}{|x - y + t(\xi/|\xi|)|^2} D_j. \tag{12}$$

Notice that the the denominators in (12) do not vanish when $x \in U$, $y \in \text{supp}(f)$ and $\xi \in \text{supp}(\psi)$ (see (8).) I denote the D_j-coefficient in (12) by $a_j(x, y, \xi)$. It is a homogeneous of degree 0 in ξ function that depends on x and y smoothly. The formula (11) implies

$$Le^{[(x-y)\xi+t|\xi|]} = e^{[(x-y)\xi+t|\xi|]}.$$

Therefore,

$$\begin{aligned} u_+^1(t, x) &= \int\int \frac{\psi(\xi)\chi(\xi)}{|\xi|} L^k \big(e^{i[(x-y)\xi+t|\xi|]}\big) f(y) dy d\xi \\ &= \int\int (L^t)^k \left(\frac{\psi(\xi)\chi(\xi)}{|\xi|}\right) e^{i[(x-y)\xi+t|\xi|]} f(y) dy d\xi \end{aligned} \qquad (13)$$

where

$$L^t = -\sum_{j=1}^{n} a_j(x, y, \xi) D_j + b(x, y, \xi)$$

with

$$b(x, y, \xi) = -\frac{1}{i} \sum_{j=1}^{n} \frac{\partial a_j(x, y, \xi)}{\partial \xi_j}.$$

The number k in (13) is arbitrary.

Reminder. Let z be any complex number. A function $h(\xi)$ defined on $\mathbf{R}^n \setminus \{0\}$ is colled homogeneous of degree z if

$$h(\tau\xi) = \tau^z h(\xi), \quad \tau > 0. \qquad (14)$$

A homogeneous function of degree z is completely determined by its values on the unit sphere in $\mathbf{R}^n$. We will use the following fact: *a partial derivative of a homogeneous function of degree z is a homogeneous function of degree $z - 1$.* To see that, one differentiates both sides of (14) with respect to ξ_j. We say that $h(\xi)$ is homogeneous of degree z when $|\xi| \geq R$ is (14) holds for when $|\xi| \geq R$ and $\tau \geq 1$.

As we have already noticed, the coefficients a_j are homogeneuos in ξ of degree 0. The free term $b(x, y, \xi)$ in (13) is ξ-homogeneous of degree -1.
Exercise. Show that

$$(L^t)^k = \sum_{|\alpha| \leq k} a_\alpha(x, y, \xi) D^\alpha$$

where the coefficient $a_\alpha(x, y, \xi)$ is ξ-homogeneous of degree $|\alpha| - k$.
In (13), the operator $(L^t)^k$ is applied to the function $\chi(\xi)\psi(\xi)/|\xi|$, which is homogeneous of degree -1 when $|\xi| \geq 1$. The function $D^\alpha(\chi(\xi\psi(\xi)/|\xi|)$ is homogeneous of degree $-1 - |\alpha|$ when $|\xi| \geq 1$, so $(L^t)^k(\chi(\xi)\psi(\xi)/|\xi|)$ is homogeneous of degree $-k - 1$ when

$|\xi| \geq 1$. Differentiating this function with respect to x and y does not change its degree of homogeneity in ξ. Therefore, for any multi-index β,

$$|D_x^\beta((L^t)^k(\chi(\xi)\psi(\xi)/|\xi|))| \leq C_\beta(1+|\xi|)^{-k-1} \tag{15}$$

when $x \in U$, $y \in \mathrm{supp}(f)$, and $\xi \in \mathrm{supp}(\psi)$. The function $\hat{f}(\xi)$ is bounded as the Fourier transform of an L^1-function, so, by taking $k = n + N$, we make sure that the integral (13)can be differentiated up to N times with respect to x. The number N is arbitrary. Therefore, $u_+^1(x) \in C^\infty(U)$. In particular, $x_0 \notin \mathrm{sing\ supp}(u(t))$.
Q.E.D.
Corollary 2.

$$\mathrm{sing\ supp}(u_+(t)) \subset \mathrm{sing\ supp}(f) - t\Sigma_1(f). \tag{16}$$

Proof. Suppose that $x_0 \notin \mathrm{sing\ supp}(f) - t\Sigma_1(f)$. Than there exists a neighborhood U of $\mathrm{sing\ supp}(f)$ such that $x_0 \notin U - t\Sigma_1(f)$. Let $\phi(x) \in C_0^\infty(U)$ and $\phi(x) = 1$ in a neighborhood of $\mathrm{sing\ supp}(f)$. The function $u_+(t)$ is the sum of $u_+'(t)$ and $u_+''(t)$ where $u_+'(t)$ corresponds to the initial condition $\phi(x)f(x)$ and $u_+''(x)$ corresponds to the initial condition $(1-\phi(x))f(x)$. Notice that $(1-\phi(x)f(x) \in C_0^\infty$, so $u_+''(t) \in C^\infty$. Then, $x_0 \notin \mathrm{supp}(\phi f) - t\Sigma_1(\phi f)$ because $\sigma(\phi f) \subset \Sigma(f)$. By Lemma 1, $x_0 \notin \mathrm{sing\ supp} u_+'(t)$.
Q.E.D.

Now, we are ready to formulate the "propagation of singularities theorem."
Theorem. *Let $u(t)$ be the solution of the problem (1) at the moment t. Then*

$$\mathrm{sing\ supp}(u(t)) \subset S_+(t) \cup S_-(t) \tag{17}$$

where

$$S_\pm(t) = \cup_{(x,\xi)\in WF(f)}(x \mp t(\xi/|\xi|)). \tag{18}$$

The Theorem has a simple interpretation. For a point $x_0 \in \mathrm{sing\ supp}(f)$, draw all the rays that emanate from x_0 and that have ξ as its direction vector where $(x_0, \pm\xi) \in WF(f)$. Let $S_{x_0}(t)$ be the union of points that lie on these rays and that are at the distance t from x_0. Then

$$\mathrm{sing\ supp}(u(t)) \subset \cup_{x_0 \in \mathrm{sing\ supp}(f)} S_{x_0}(t).$$

In short, this means that the singularities of the wave equation are propagated with the velocity ± 1 in the directions of the wave front set of the initial data. One can prove a sharper version of the Theorem that gives a localization for the wave front set of $u(t)$, not just for the $\mathrm{sing\ supp} u(t)$.
Proof of the Theorem. First, I assume that $f(x)$ is an L^2 function with compact support. Suppose that $x_0 \notin S_+(t)$. Then $x \notin y - t(\Sigma_y(f) \cap S^{n-1})$ for every $y \in \mathrm{supp}(f)$. Here, as usual, S^{n-1} is the unit sphere. By Proposition 3 from the notes on the wave front set, there exists a neighborhood U_y of the point y such that

$$x_0 \notin U_y - t\Sigma_1(\phi f) \tag{19}$$

for every function $\phi \in C_0^\infty(U_y)$. The neighborhoods U_y cover $\mathrm{supp}(f)$. We have assumed that $\mathrm{supp}(f)$ is compact, so one can find a finite number of them, $U_1, \ldots, U_p$, that still cover supp$f$. Let $\{\phi_j\}$ be a partition of unity that corresponds to the covering $\{U_j\}$: functions $\phi_j(x) \in C_0^\infty(U_j)$ and their sum equals 1 in a neighborhood of supp(f). Then the solution of the problem (1) is the sum of the functions $u_j(t)$ that solve that problem with $f(x)$ replaced by $\phi_j(x)f(x)$. Let $u_{j,\pm}$ be the corresponding "half-solutions". By Lemma 1, (19) implies that $x_0 \notin$ sing supp$(u_{j,+}(t))$ for all j. Therefore, $x_0 \notin$ sing supp$(u_+(t))$. By replacing $t \mapsto -t$, one concludes that $x_0 \notin S_-(t)$ implies $x_0 \notin$ sing supp$(u_-(t))$. This proves the Theorem in the case when f is an L^2 function with compact support.

One can easily remove the assumption of f having a compact support. To do that, one recalls from the standard PDE course that solutions of the wave equation are propagated with speed 1, that is

$$\operatorname{supp}(u(t)) \subset \{x : \operatorname{dist}(x, \operatorname{supp} f) \leq |t|\}. \tag{20}$$

Suppose that the support of f is not compact. Let x_0 be a point. We would like to find out whether $x_0 \in$ sing supp$(u(t))$. Take a function $\psi(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\psi(x) = 1$ when $|x - x_0| \leq |t| + 1$. Let $\tilde{u}(t)$ be the solution of the problem (1), with $f(x)$ replaced by $\psi(x)f(x)$. The relation (20) implies $u(x,t) = \tilde{u}(x,t)$ when $|x - x_0| < 1$. One notices that $WF(\psi f) \subset WF(f)$ and applies the Theorem to $\tilde{u}(t)$.

Let f be a distribution with compact support. Then one can represent f in the form $f = (1 - \Delta)^k g$ where $g(x) \in L^2(\mathbf{R}^n)$. In fact, the Fourier transform of the function f satisfies the estimate

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^m$$

for some number m. We take

$$g(x) = (2\pi)^{-n} \int e^{ix\xi} \frac{\hat{f}(\xi)}{(1 + |\xi|^2)^k} d\xi \tag{21}$$

where k is chosen in such a way that $2k - m > n/2$ (that guarantees $\hat{f}(\xi)/(1 + |\xi|^2)^k \in L^2$.) We will prove later that $WF(g) \subset WF(f)$ (the function g is a pseudodifferential operator applied to f, and, as we will see, this is a general property of pseudodifferential operators.) Let $v(t,x)$ be the solution of the problem (1), with f replaced by $g(x)$. Then $u(t) = (1 - \Delta)^k u(t)$, so sing supp$u(t) \subset$ sing supp$v(t)$. Now, we can apply the theorem to $v(t)$ and derive the statement about $u(t)$ from that.
Q.E.D.