The wave front set of a distribution

The Fourier transform of a smooth compactly supported function $u(x)$ decays faster than any negative power of the dual variable ξ; that is for every number N there exists a constant C_N such that

$$|\hat{u}(\xi)| \le C_N(1+|\xi|)^{-N}. \tag{1}$$

On the other hand, if the Fourier transform of a distribution with compact support satisfies the estimate (1) then this distribution is actually induced by a smooth function. Therefore, the estimate (1) can be viewed as a characteristic property for smoothness. The singular support of a distribution tells us where the singularities of a distribution lie. The wave front set gives more precise description of singularities; it tells us not only at what points a singularity occur, but it also indicates the directions in the dual space from which the singularities are coming; that is, in what directions the estimate (1) does not hold.

Let us start with some definitions. A set $V \subset \mathbf{R}^n \setminus \{0\}$ is called *a conic set* if, together with any point ξ, it contains all the points $t\xi$ where $t > 0$. A conic set is completely determined by its intersection with the unit sphere S^{n-1} in $\mathbf{R}^n$. By a conic neighborhood of a point $\xi \in \mathbf{R}^n \setminus \{0\}$ we mean an open conic set that contains ξ.

Let $u \in \mathcal{E}'$ be a distribution in $\mathbf{R}^n$ with compact support. Its Fourier transform is a smooth function. We define the set $\Sigma(u) \in \mathbf{R}^n \setminus \{0\}$ by saying that $\xi \notin \Sigma(u)$ if there exists a conic neighborhood V of ξ such that the estimate (1) holds in V for all N. It follows immediately from the definition that $\Sigma(u)$ is a closed conic set. Notice that the distribution u is induced by a smooth function if and only if $\Sigma(u) = \emptyset$.

Proposition 1. *Let $u \in \mathcal{E}'(\mathbf{R}^n)$ and $\phi \in C^\infty(\mathbf{R}^n)$. Then $\Sigma(\phi u) \subset \Sigma(u)$.*

Proof. First, let $\phi \in C_0^\infty$. Let $\xi \notin \Sigma(u)$, and let V be a conic neighborhood of ξ where the estimate (1) holds. We take a smaller conic neighborhood, V' of ξ the closure of which lies in V, and we will prove the estimate (1) for the product ϕu in V'.

First, there exists a constant c such that, for every $\eta \in V'$, the closed ball of radius $c|\eta|$, centered at η, lies in V. In fact, the distance between $V' \cap S^{n-1}$ and the complement of V is positive. Choose c to be a positive number that is smaller that that distance. The inequality $|\zeta - \eta| < c|\eta|$ implies $|(\zeta/|\eta|) - (\eta/|\eta|)| < c$; so $(\zeta/|\eta|) \in V'$, and $\zeta \in V'$ because V' is conic.

The Fourier transform of ϕu equals the convolution of Fourier transforms,

$$\widehat{\phi u}(\eta) = (2\pi)^{-n} \int \hat{\phi}(\eta - \zeta)\hat{u}(\zeta)d\zeta. \tag{2}$$

The Fourier transform of u has an upper bound

$$|\hat{u}(\zeta)| \le C(1+|\zeta|)^M \tag{3}$$

for some number M. The integral in (2) can be broken into

$$I_1 + I_2 = \int_{|\zeta-\eta|\le c|\eta|} + \int_{|\zeta-\eta|>c|\eta|}.$$

In the first integral, $\zeta \in V$, so

$$|\hat{u}(\zeta)| \le C_N(1+|\zeta|)^{-N} \le C'_N(1+|\eta)^{-N}$$

because $|\zeta| \ge (1-c)|\eta|$. Therefore,

$$|I_1| \le C'_N(1+|\eta|)^{-N} \int |\hat{\phi}(\eta)| d\eta. \tag{4}$$

To estimate the integral I_2, we notice that $|\zeta - \eta| \ge c|\eta|$ implies

$$|\zeta - \eta| \ge |\zeta| - |\eta| \ge |\zeta| - \frac{1}{c}|\zeta - \eta|,$$

and, therefore,

$$|\zeta - \eta| \ge \frac{c}{c+1}|\zeta|. \tag{5}$$

The estimates (1) and (3) imply that, for any choice of N,

$$|I_2| \le C \int_{|\zeta-\eta|\ge c|\eta|} (1+|\zeta-\eta|)^{-N-M-n-1}(1+|\zeta|)^M d\zeta.$$

The integrand in the last formula is bounded by

$$(1+|\zeta-\eta|)^{-N}(1+|\zeta-\eta|)^{-M-n-1}(1+|\zeta|)^M \le C_1(1+|\eta|)^{-N}(1+|\zeta|)^{-n-1}$$

when $|\zeta - \eta| \ge c|\eta|$ (see (5).) Therefore,

$$|I_2| \le C_2(1+|\eta|)^{-N}.$$

The last estimate, together with (4), implies

$$|\widehat{\phi u}(\eta)| \le C_3(1+|\eta|)^{-N}.$$

This proves the Proposition in the case $\phi \in C_0^\infty$. If a function ϕ is not compactly supported then one can find a function $\phi' \in C_0^\infty$ that coincides with ϕ in a neighborhood of supp$u$. Clearly, $\phi u = \phi' u$. Q.E.D.

Corrolary 2. *Let $u \in \mathcal{D}'(\mathbf{R}^n)$, and let $\phi_1, \phi_2 \in C_0^\infty(\mathbf{R}^n)$. Suppose that $\phi_2(x) \ne 0$ when $x \in \mathrm{supp}(\phi_1)$. Then $\Sigma(\phi_1 u) \subset \Sigma(\phi_2 u)$.*

Proof. Let U be a neighborhood of $\mathrm{supp}(\phi_1)$ such that $\phi_2(x) \ne 0$ when $x \in U$, and let V be a smaller neighborhood of $\mathrm{supp}(\phi_1)$:

$$\mathrm{supp}(\phi_1) \subset V; \quad \bar{V} \subset U.$$

Let $\chi(x)$ be a smooth function such that $\chi(x) = 1$ when $x \in V$ and $\chi(x) = 0$ when $x \notin U$. We define a function

$$\psi(x) = \begin{cases} \chi(x)/\phi_2(x), & \text{if } x \in U; \\ 0, & \text{if } x \notin U. \end{cases}$$

Clearly, $\psi(x)$ is a smooth, compactly supported function, and $\phi_1 = \psi(\phi_2 u)$.
Q.E.D.

Let Ω be an open set in $\mathbf{R}^n$, and let $u \in \mathcal{D}(\Omega)$. For a point $x \in \Omega$, we define

$$\Sigma_x(u) = \cap \Sigma(\phi u); \quad \phi \in C_0^\infty(\Omega), \quad \phi(x) \neq 0.$$

As an intersection of closed conic sets, $\Sigma_x(u)$ is a closed conic set.

Proposition 3. *Let Γ be a conic neighborhood of $\Sigma_x(u)$, $u \in \mathcal{D}'(\Omega)$. Then there exists a neighborhood U of x such that $\Sigma(\phi u) \in \Gamma$ for every function $\phi(x) \in C_0^\infty(U)$.*

Proof. The set $K = S^{n-1} \setminus \Gamma$ is a closed subset of the unit sphere. For every point $\omega \in K$ there exists a function $\phi_\omega \in C_0^\infty(\Omega)$ such that $\phi_\omega(x) \neq 0$ and a neighborhood of ω does not intersect with $\Sigma(\phi_\omega u)$. These neighborhoods cover K. One can find a finite number of them that still cover K. Therefore, there exists a finite number of functions ϕ_j $C_0^\infty(\Omega)$ such that $\phi_j(x) \neq 0$ and $K \cap (\cap_j \Sigma(\phi_j u)) = \emptyset$. Because Γ is a conic set, we conclude

$$\cap_j \Sigma(\phi_j u) \subset \Gamma.$$

Let U be a neighborhood of x that all ϕ_j's do not vanish in U. By Corrolary 2, $\Sigma(\phi u) \subset \Sigma(\phi_j u)$ for every function $\phi \in C_0^\infty(U)$. Therefore, $\Sigma(\phi u) \subset \Gamma$.
Q.E.D.

One can interpret Proposition 3 in the following way: $\Sigma_x(u)$ is the limit of $\Sigma(\phi u)$ when $\mathrm{supp}(\phi) \to \{x\}$ and $\phi(x) \neq 0$. Now, we are ready to define the wave front set of a distribution.

Definition. *The wave front set of a distribution $u \in \mathcal{D}(\Omega)$ is defined as*

$$WF(u) = \{(x, \xi) \in \Omega \times (\mathbf{R}^n \setminus \{0\}) : \xi \in \Sigma_x(u)\}.$$

It is a simple exercise to derive from the definition of the wave front set and from Proposition 3 that the projection of $WF(u)$ on Ω is exactly the singular support of u.

Example 4. Let $P_k \in \mathbf{R}^n$ be the k-dimensional co-ordinate plane $x_{k+1} = \cdots = x_n = 0$. By x' I will denote the collection $(x_1, \ldots, x_k)$, and x'' is the collection of remaining co-ordinates, so $x = (x', x'')$. For a function $u(x') \in C^\infty(P_k)$, we will compute the wave front set of the distribution $u(x')\delta(x'')$. This distribution acts on a test function in the following way

$$\langle u(x')\delta(x''), \psi \rangle = \int u(x')\psi(x', 0)dx'.$$

The support of this distribution is $\{x = (x', 0) : x' \in \mathrm{supp}(u)\}$. Choose a point $x_0 = (x_0', 0)$ from this set. Let ϕ be a compactly supported smooth function such that $\phi(x_0) \neq 0$. The Fourier transform of the distribution $\phi u(x')\delta(x'')$ equals

$$F(\xi', \xi'') = \int u(x')\phi(x', 0)e^{-x'\xi'}dx'.$$

Let $\Gamma_k = \{(\xi', \xi'') \neq 0 : \xi' = 0\}$. On the whole cone Γ_k, the function $F(\xi)$ is constant (the integral of $u(x')\phi(x', 0)$.) For every neighborhood of x_0, one can find a function ϕ

supported in that neighborhood such that the integral of $u(x')\phi(x',0)$ does not vanish. So, by Proposition 3, $\Gamma_k \subset \Sigma_{x_0}(u\delta(x''))$. On the other hand, if $\xi_0 \notin \Gamma_k$ then $|\xi''| \leq C|\xi'|$ for every point $\xi = (\xi', \xi'')$ from a certain conic neighborhood Γ of ξ_0. Therefore, for every N,

$$|F(\xi)| \leq C_N(1+|\xi'|)^{-N} \leq C'_N(1+|\xi|)^{-N}$$

when $\xi \in \Gamma$, and $\xi_0 \notin \Sigma_{x_0}(u\delta(x''))$. We conclude that

$$WF(u(x')\delta(x'')) = \{(x', x''; \xi', \xi'') : x' \in \operatorname{supp}(u), \xi' = 0\}. \tag{6}$$

Transformation of the wave front set under a diffeomorphism.

Let $\Phi : \Omega \to \Omega'$ be a diffeomorphism, and let $u \in \mathcal{D}'(\Omega')$. The distribution $v = \Phi^* u$ acts according to the formula

$$\langle v, \phi \rangle = \langle u, |\Psi'(y)|\phi(\Psi(y)) \rangle$$

where $\Psi = \Phi^{-1}$ and $|\Psi'|$ is the absolute value of the Jacobian of Ψ. In particular, if u is a distribution with compact support then

$$\hat{v}(\xi) = \langle u, |\Psi'(y)|e^{-i\Psi(y)\cdot\xi} \rangle = \langle u, a(y)|\Psi'(y)|e^{-i\Psi(y)\cdot\xi} \rangle$$

where $a(y)$ is a smooth compactly supported function that equals 1 identically on $\operatorname{supp}(u)$. The support of $a(y)$ can be made as close to $\operatorname{supp}(u)$ as one wishes.To simplify notations, we set

$$b(y) = a(y)|\Psi'(y)|.$$

By the definition of the Fourier transform of a distribution,

$$\begin{aligned} \hat{v}(\xi) &= \langle \hat{u}, \mathcal{F}^{-1}\big(b(y)e^{-i\Psi(y)\cdot\xi}\big) \rangle \\ &= (2\pi)^{-n} \int\int \hat{u}(\eta)b(y)e^{i(y\eta - \Psi(y)\cdot\xi)}dyd\eta. \end{aligned} \tag{7}$$

Fix a point $x_0 \in \Omega$. We will assume that the support of u lies in a ball of sufficiently small radius centered at the point $y_0 = \Phi(x_0)$. We also assume that the support of $b(y)$ also lies in that ball. To make notations simpler, set $x_0 = y_0 = 0$.

Lemma 5. *Let $A(y) = \Psi'(y)$ be the Jacobi matrix of Ψ. Then*

$$\Sigma(\Phi^* u) \subset \bigcup_{y \in \operatorname{supp}(u)} (A^t(y))^{-1}\Sigma(u). \tag{8}$$

Proof. Let ξ_0 be a point that does not belong to $\cup_{y\in \operatorname{supp} u}(A^t(y))^{-1}\Sigma(u)$. Then there exists a conic neighborhood Γ of ξ_0 and a conic neighborhood V of $\Sigma(u)$ such that

$A^t(y)\Gamma \cap V = \emptyset$ when $y \in \mathrm{supp} u$. Let Γ' be a smaller conic neighborhood of ξ_0, $\bar{\Gamma}' \subset \Gamma$. We break the integral (7) into the sum

$$v_1(\xi) + v_2(\xi) = (2\pi)^{-n} \int_{\eta \in V} + (2\pi)^{-n} \int_{\eta \notin V}.$$

It is easy to estimate $v_2(\xi)$. The function $\hat{u}(\eta)$ decays rapidly outside of V, so

$$(2\pi)^{-n} b(y) \int_{V^c} \hat{u}(\eta) e^{iy\eta} d\eta$$

is a smooth, compactly supported function. Therefore, after having made a substitution $x = \Psi(y)$, one would recognize $v_2(\xi)$ as the Fourier transform of a smooth, compactly supported function. Thus, $v_2(\xi)$ decays rapidly.

To estimate $v_1(\xi)$ we will use multiple partial integrations. Notice that

$$\begin{aligned} D_j e^{i(y\eta - \Psi(y)\cdot\xi)} &= \left(\eta_j - \sum_k \frac{\partial \Psi_k(y)}{\partial y_j} \xi_k \right) e^{i(y\eta - \Psi(y)\cdot\xi)} \\ &= (\eta - A^t(y)\xi)_j e^{i(y\eta - \Psi(y)\cdot\xi)}. \end{aligned}$$

We introduce a first order differential operator

$$L = \sum_j \frac{(\eta - A^t(y)\xi)_j}{|\eta - A^t(y)\xi|^2} D_j. \tag{9}$$

The denominator in (9) does not vanish when $\xi \in \Gamma$ and $\eta \in V$. One has

$$L e^{i(y\eta - \Psi(y)\cdot\xi)} = e^{i(y\eta - \Psi(y)\cdot\xi)}. \tag{10}$$

Let $\xi \in \Gamma'$ and $|\xi| \geq 1$. Then

$$\begin{aligned} v_1(\xi) &= (2\pi)^{-n} \int_V \int \hat{u}(\eta) b(y) L^k e^{i(y\eta - \Psi(y)\cdot\xi)} dy d\eta \\ &= (2\pi)^{-n} \int_V \int \hat{u}(\eta) ((L^t)^k b(y)) e^{i(y\eta - \Psi(y)\cdot\xi)} dy d\eta \end{aligned} \tag{11}$$

where

$$L^t = -\sum_j D_j \frac{(\eta - A^t(y)\xi)_j}{|\eta - A^t(y)\xi|^2}.$$

The operator $(L^t)^k$ is a differential operator of order k in y.
Exercise. By induction, show that the coefficients of $(L^t)^k$ are of the form

$$\sum_{m=0}^{k} \frac{P_{k+m}(y, \xi, \eta)}{|\eta - A^t(y)\xi|^{2(k+m)}}$$

where P_{k+m} is a polynomial in (ξ,η) of degree $k+m$ with smooth in y coefficients.

The cones Γ and $\cup(A^t(y)^{-1}V$ do not intersect, and the closure of Γ' lies in Γ, so

$$|\eta - A^t(y)\xi| \geq C|\eta|, \quad |\eta - A^t(y)\xi| \geq C|\xi|$$

when $\xi \in \Gamma'$ and $\eta \in V$. If one assumes in addition that $|\xi| \geq 1$ then

$$|\eta - A^t(y)\xi| \geq C(1+|\xi|+|\eta|).$$

The last estimate, the result of the exercise, and (11) imply

$$|v_1(\xi)| \leq C_k \int_V |\hat{u}(\eta)|(1+|\xi|+|\eta|)^{-k} d\eta$$

for any k. On the other hand,

$$|\hat{u}(\eta)| \leq C(1+|\eta|)^M$$

for some M because u is a distribution with compact support. By choosing $k = N+M+n+1$, one gets the desired estimate

$$|v_1(\xi)| \leq C(1+|\xi|)^{-N}.$$

Q.E.D.

Now, we can formulate the theorem that says how the wave front set of a distribution is transformed under a change of variables.

Theorem 6. *Let Ω and Ω' be open domains in $\mathbf{R}^n$, and let $\Phi : \Omega \to \Omega'$ be a diffeomorphism. The wave front set of the pull-back $\Phi^* u$ of a distribution $u \in \mathcal{D}'(\Omega')$ is given by the following formula*

$$WF(\Phi^* u) = \{(x,\xi) \in \Omega \times (\mathbf{R}^n \setminus \{0\}) : (\Phi(x), (\Phi'(x))^t \xi) \in WF(u)\}. \tag{12}$$

Before we prove theorem 6, let us discuss how to interpret it. The pull-back of a one-form $\eta = \sum \eta_j dy_j$ on Ω' is

$$\Phi^*\eta = \sum_{k=1}^n \xi_k dx_k = \sum_{k=1}^n \sum_{j=1}^n \frac{\partial \Phi_j}{\partial x_j} \eta_j dx_k;$$

so

$$\xi_k = \sum_{j=1}^n \frac{\partial \Phi_j}{x_k} \eta_j.$$

The last equation can be re-written in the matrix form as

$$\xi = (\Phi'(x))^t \eta. \tag{13}$$

A diffeomorphism $\Phi : \Omega \to \Omega$ gives rise to a mapping $\hat{\Phi} : T^*(\Omega') \to T^*(\Omega)$:

$$\hat{\Phi}(y, \eta) = (\Phi^{-1}(y), (\Phi'(\Phi^{-1}(y))^t)\eta. \tag{14}$$

This mapping is a diffeomorphism, and it maps the zero section $\Omega' \times \{0\}$ onto the zero section $\Omega \times \{0\}$. Theorem 6 says that $WF(\Phi^* u) = \hat{\Phi}(WF(u))$. In other words, the wave front set is a correctly defined set in the cotangent bundle over a domain. One should think of $\Omega \times (\mathbf{R}^n \setminus \{0\})$ as the space of non-zero covectors over Ω.

Proof of Theorem 6. We start from proving the inclusion

$$WF(\Phi^* u) \subset \hat{\Phi}(WF(u)). \tag{15}$$

Suppose that

$$(y, \eta) = \hat{\Phi}^{-1}((x, \xi)) \notin WF(u).$$

By Proposition 3, there exists a conic neighborhood Γ' of η and a neighborhood V of the point y such that

$$\Sigma(\psi u) \cap \Gamma' = \emptyset \tag{16}$$

for every smooth function ψ that is supported in V. Let $U = \Phi^{-1}(V)$; this is a neighborhood of the point x. The equation (16) implies

$$\xi \notin (\Phi'(x)^t)(\Sigma(\psi u)),$$

and, therefore,

$$\xi \notin (\Phi'(x')^t)(\Sigma(\psi u)) \tag{17}$$

for every point x' from a sufficiently small neighborhood U' of the point x. Let $\tilde{U} = U \cap U'$ and $\tilde{V} = \Phi(\tilde{U})$. Take any function $\phi \in C_0^\infty(\tilde{U})$ such that $\phi(x) \neq 0$. Let $\psi(z) = \phi(\Phi^{-1}(z))$. Then $\phi\Phi^* u = \Phi^*(\psi u)$, and by Lemma 5, $\xi \notin \Sigma(\phi\Phi^* u)$ (see (17).) By the definition of the wave front set, $(x, \xi) \notin WF(\Phi^* u)$, so the inclusion (15) has been established.

To prove the opposite inclusion, we notice that the composition of $\hat{\Phi}$ and $\widehat{\Phi^{-1}}$ is the identity mapping and $(\Phi^{-1})^* \Phi^* u = u$; so the inclusion (15) written for Φ^{-1} is equivalent to

$$WF(\Phi^* u) \supset \hat{\Phi}(WF(u)).$$

Q.E.D.

Example 7. Let M be a k-dimensional smooth submanifold in $\mathbf{R}^n$. For a function $u \in C^\infty(M)$ we define the distribution $u\delta_M \in \mathcal{D}'(\mathbf{R}^n)$:

$$\langle u\delta_M, \phi \rangle = \int_M u(x)\phi(x)dS$$

where dS is the area element in M that is induced by the Euclidean metric in $\mathbf{R}^n$. Let us compute $WF(u\delta_M)$. First, a point (x, ξ) may lie in $WF(u\delta_M)$ only if $x \in M$ and

$x_0 \in \text{supp}(u)$. Theorem 6 allows us to use any co-ordinate system for computing the wave front set. Let $x \in \text{supp}(u) \subset M$. First, we take an orthogonal co-ordinate system $(y_1, \dots, y_n)$, with the origin at the point x_0, with the first k co-ordinate axes lying in the tangent plane $T_{x_0}(M)$ to M at the point x_0, and the last $n-k$ co-ordinate axes going along its orthogonal complement. The transition from the old co-ordinate system $(x_1, \dots, x_n)$ to the new one is given by the composition of a shift and an orthogonal transformation. A neighborhood of the point x_0 in M is given by equation

$$y_l = F_l(y_1, \dots, y_k), \quad l = k+1, \dots, n,$$

with smooth functions F_l. Notice that

$$\nabla F_l(0) = 0, \quad l = k+1, \dots, n. \tag{18}$$

In a neighborhood of the point x_0 in $\mathbf{R}^n$, we introduce co-ordinates $(z_1, \dots, z_n)$:

$$z_j = \begin{cases} y_j, & \text{when } j \le k; \\ y_j - F_j(y_1, \dots, y_k), & \text{when } j > k. \end{cases} \tag{19}$$

Equations (18) imply that the Jacobi matrix $(\partial z/\partial y)(0)$ is the identity matrix, so (19) define a diffeomorphism in a neighborhood of x_0. In that neighborhood of x_0, the manifold M is given by the equations $z_{k+1} = \cdots = z_n = 0$, and the distribution $u\delta_M$ acts on a function that is supported in a neighborhood of x_0 by the formula

$$\langle u, \delta_M \phi \rangle = \int u(z')\phi(z', 0)m(z')dz'$$

where $z' = (z_1, \dots, z_k)$, $z'' = (z_{k+1}, \dots, z_n)$, and $m(z')dz'$ is the area element dS written in the local co-ordinates z' on M. From Example 4, we know that $(x_0, \zeta) \in WF(u\delta_M)$ when $\zeta' = 0$. If interpreted as a point in the cotangent space $T^*_{x_0}(M)$, it annihilates the tangent space $T_{x_0}(M)$. A subspace in $T^*_{x_0}(M)$ that annihilates $T_{x_0}(M)$ is called the *normal space* to M at x_0, and it is denoted by $N_{x_0}(M)$. In the original co-ordinates, it is given by

$$N_{x_0}(M) = \{\xi \in \mathbf{R}^n : \xi \perp T_{x_0}(M)\}.$$

We conclude that

$$WF(u\delta_M) = \{(x, \xi) : x \in \text{supp}(u) \subset M, \xi \in N_x(M) \setminus \{0\}\}.$$