Problems for "Introduction to micro-local Analysis"

1. In the space of compactly supported smooth functions in $\mathbf{R}^n$, the base of open neighborhoods of a function $\psi(x)$ is defined as a collection of sets

$$U_{\gamma(x),\alpha}(\psi) = \{\phi(x) \in C_0^\infty(\mathbf{R}^n) : |D^\alpha(\phi(x) - \psi(x))| < \gamma(x)\}$$

for all continuous functions $\gamma(x) > 0$ and all multi-indices α. Show that the resulting topology induces the following notion of convergence: $\phi_n \to \phi$ if all functions ϕ_n are supported in the same bounded region and $D^\alpha \phi_n$ converges to $D^\alpha \phi$ uniformly for every multi-index α.

2. Let $\phi(x) \in C_0^\infty(\mathbf{R}^n)$, and let s be a non-negative real number. Investigate the asymptotic behavior of the integral

$$I_\phi(\epsilon) = \int_{|x|\geq\epsilon} \frac{\phi(x)}{|x|^{n+s}} dx$$

as $\epsilon \to 0$. Use the asymptotics of $I_\phi(\epsilon)$ for regularizing the function $|x|^{-n-s}$. The answer depends on whether the number s is integer or not.

3. Prove that the Fourier transform is a continuous map from the Schwarz space $\mathcal{S}(\mathbf{R}^n)$ onto itself.

4. Let $u \in \mathcal{S}'(\mathbf{R}^n)$ and $\phi(x) \in \mathcal{S}(\mathbf{R}^n)$. Prove that the Fourier transform of $\phi * u$ equals the product of Fourier transforms of ϕ and u.

5. Find Fourier transforms of the following distributions:

a) P.V$(1/x)$;

b) $(x \pm i0)^{-1}$;

c) x_+^λ.

6. Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary. Find the wave front set of the characteristic function χ_Ω of the domain Ω. Do the same problem for

$$\Omega = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_j \geq 0, j = 1, \ldots, n\}.$$