Global Differential Geometry HW 3

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1) 4-2a,4-5
2) 5-1
3) Consider the surface of revolution defined by

$$(s,t) \to \left(s\cos t, s\sin t, s^2\right).$$

Compute the components of its Christoffel symbols. Show that the meridian curves $t = t_0$ are geodesics (Hint: the curves look like $\gamma(\tau) = (\beta(\tau), t_0)$ but must be parametrized by arclength, which tells you what β' must be. Then use the fact that geodesics satisfy $\nabla_{\dot\gamma}\dot\gamma = 0$.)

4) In this exercise, we consider the three-dimensional Heisenberg group (also called Nil). Consider the Lie group consisting of unit upper triangular matrices with coordinate (x,y,z) given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \to (x,y,z).$$

The group multiplication is given in the usual way. The tangent space at the identity (also called the Lie algebra) consists of matrices of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the tangent space at the identity. Give an expression for the extension of these vectors to global vector fields by the relationship

$$E_i(x,y,z) = \left(L_{(x,y,z)}\right)_* E_i,$$

where L_γ denotes left multiplication by γ. Also compute the Lie brackets of these vector fields. (They should agree with the usual matrix (commutator) Lie bracket on the Lie algebra.)

b) Define a Riemannian metric by

$$g = A\left(\phi^1\right)^2 + B\left(\phi^2\right)^2 + C\left(\phi^3\right)^2$$

where ϕ^i are the covector fields dual to E_i and A, B, C are positive constants. Compute the connection by computing

$$\nabla_{E_i} E_j$$

for all i, j. (Hint: Use the general formula for the connection, and use the fact that $g(E_i, E_j)$ is constant. Note that even if you failed to do part a, you can compute the Lie brackets of the matrices and that is sufficient to do this exercise.)

c) Compute the Christoffel symbols in coordinates $\{x, y, z\}$ directly. Check that they agree with your results from b.