Global Differential Geometry HW 4

David Glickenstein

October 11, 2007

1) Show that any two-dimensional manifold only has one sectional curvature, say K. Show that the Ricci and scalar curvatures are

$$\begin{aligned} \mathrm{Rc}\,(X, Y) &= K g\,(X, Y) \\ R &= 2K. \end{aligned}$$

2) Consider a surface of revolution defined as follows. Let $(f(s), 0, g(s))$ be a curve in $\mathbb{R}^3$ parametrized by arclength (so $(f')^2 + (g')^2 = 1$). Let Σ be the surface parametrized by (s, θ) gotten by rotating the curve around the z-axis, giving the parametric surface $(f(s)\cos\theta, f(s)\sin\theta, g(s))$. It is easy to see that the induced Riemannian metric on this submanifold is $ds^2 + f^2 d\theta^2$ (check it for yourself). Show that the sectional curvature is

$$K(s, \theta) = -\frac{f''(s)}{f(s)}.$$

It is not hard to show that the metric on the sphere is

$$ds^2 + \sin^2 s \; d\theta^2.$$

Thus the sectional curvature is $+1$ at any point.

3) 7.2

4) In this problem, we give a new geometric interpretations of the Ricci and scalar curvatures

a) Let B be a symmetric bilinear form on an inner product space (V, g), i.e.

$$B(x, y) = B(y, x).$$

Consider an orthonormal basis $e_1, \ldots, e_n$ of V so that if $x = x^i e_i$ then

$$B(x, x) = \sum_{i=1}^{n} \lambda_i \left(x^i\right)^2$$

for some real λ_i. Show that

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}} B\left(x,x\right) dS^{n-1} = \frac{1}{n}\sum \lambda_i$$

where $S^{n-1} = \partial B^n$ is the unit sphere, dS^{n-1} is the standard measure on the unit sphere, and ω_{n-1} is the $(n-1)$-dimensional volume of S^{n-1}.

b) Show that the scalar curvature R satisfies

$$\frac{1}{n}R\left(p\right) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \operatorname{Rc}\left(X,X\right)\ dS^{n-1}\left(X\right)$$

where $S^{n-1} \subset T_pM$ is the unit sphere in T_pM.

c) Show that the Ricci curvature

$$\frac{1}{n-1}\operatorname{Rc}\left(X,X\right) = \frac{1}{\omega_{n-2}}\int_{S^{n-2}} K\left(X,Y\right)\ dS^{n-2}\left(Y\right)$$

where S^{n-2} is the unit sphere in T_pM orthogonal to X and K is the sectional curvature.