Note Title 5/7/2008

A.1. $y'' - y = \delta(t-3)$, $y(0)=0$, $y'(0)=0$

Laplace transform on both sides.

Let $L[y](s) = F(s)$,

Then $L[y''] = s^2F(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}$

$= s^2 F(s)$

$L[\delta(t-3)] = e^{-3s}$

Read the Table in the textbook (item 35)

$e^{-as} \longleftrightarrow \delta(t-a)$

$F(s)$ $\quad$ $f(t)$

Then the equation becomes

$$s^2F(s) - F(s) = e^{-3s}$$

$$\Rightarrow F(s) = \frac{e^{-3s}}{s^2-1}$$

This function cannot be found on the table!

But, remember Laplace transform has the shifting property $L[f(t-a)] = e^{-as}\cdot L[f(t)]$,

Thus, we only need to deal with $\frac{1}{s^2-1}$, and later incorporate e^{-3s} as a shifting.

$$\text{Since } \frac{1}{s^2-1} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1}$$

$\downarrow L^{-1}$ $\quad$ $\downarrow L^{-1}$

$\frac{1}{2}e^{t}$ $\quad$ $-\frac{1}{2}e^{-t}$

$F(s)$		$f(t)$
$\frac{1}{s^2-1}$	$\longleftrightarrow$	$\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$
$e^{-3s}\cdot\frac{1}{s^2-1}$	$\longleftrightarrow$	$\begin{cases} \frac{1}{2}e^{(t-3)} - \frac{1}{2}e^{-(t-3)}, & \text{for } t \geq 3 \\ 0, & \text{for } t < 3 \end{cases}$

shifting property of Laplace transform.

Note, when a fcn is shifted to the right by k units, the blank between [0, k] needs to be filled with zero!

Finally, the solution is
$$y(t) = \begin{cases} 0 & \text{for } 0 \leq t < 3 \\ \frac{1}{2}e^{t-3} - \frac{1}{2}e^{-t+3}, & \text{for } t \geq 3 \end{cases}$$

Q4: Let $u(x,t) = X(x)T(t)$

$u_{tt} - a^2 u_{xx} = 0 \Rightarrow \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$ (a is known). $\lambda > 0$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 & (1) \\ T'' + a^2 \lambda T = 0 & (2) \end{cases}$$

$(1) \Rightarrow X = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}x$

B.C. $X(0) = X(l) = 0 \Rightarrow C_2 = 0$, $\lambda = \left(\frac{n\pi}{l}\right)^2$ (l is known)

$\Rightarrow X_n(x) = \sin\frac{n\pi}{l}x$, $n = 1, 2, \cdots$

$(2) \Rightarrow T_n(t) = A_n \sin a\sqrt{\lambda}x + B_n \cos a\sqrt{\lambda}x$

$= A_n \sin\frac{an\pi}{l}t + B_n \cos\frac{an\pi}{l}t$, $n = 1, 2, \cdots$

$\Rightarrow u_n(x,t) = X_n(x) T_n(t) = \left(A_n \sin\frac{an\pi}{l}t + B_n \cos\frac{an\pi}{l}t\right) \sin\frac{n\pi}{l}x$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin\frac{an\pi}{l}t + B_n \cos\frac{an\pi}{l}t\right) \sin\frac{n\pi}{l}x$

Apply I.C. $\varphi(x) = u|_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$

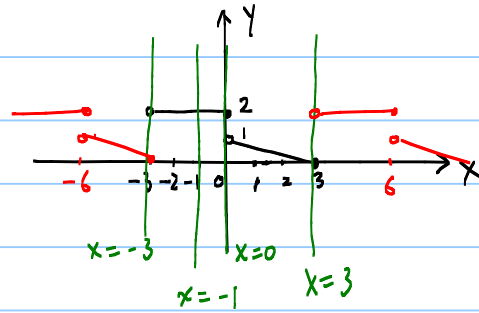
$\psi(x) = u_t|_{t=0} = \sum_{n=1}^{\infty} \frac{an\pi}{l} A_n \sin\frac{n\pi}{l}x$

$\Rightarrow B_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi}{l}x \, dx$

$\frac{an\pi}{l} A_n = \frac{2}{l}\int_0^l \psi(x) \sin\frac{n\pi}{l}x \, dx \Rightarrow A_n = \frac{2}{an\pi}\int_0^l \psi(x) \sin\frac{n\pi x}{l} dx$

A.3.

To see the value of convergence of Fourier series, especially at boundaries like $x = 3$ or $x = -3$, lets extend the graph to more than one periods.



- At $x = 0$, there's a discontinuity, thus the Fouries converges to the average of 2 and 1, which is $\frac{2+1}{2} = \boxed{\frac{3}{2}}$
- Similarity, at $x = -3$ and $x = 3$, there are discontinuities, and at both $x=3$ and $x=-3$, the Fouries converges to the average of 2 and 0, which is $\frac{2+0}{2} = \boxed{1}$
- At $x = -1$, the fcn is continuous, thus, the Fourier series converges to the fcn value at that point, which is $\boxed{2.}$

Q 4 $\quad L=2$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$=\int_0^1 x\sin\frac{n\pi}{2}x\,dx+\int_1^2(2-x)\sin\frac{n\pi}{2}x\,dx$$

$$=\int_0^1\left(-\frac{2}{n\pi}\right)\cdot x\,d\cos\frac{n\pi}{2}x+\left(-\frac{4}{n\pi}\cos\frac{n\pi}{2}x\right)\Big|_1^2+\int_1^2\frac{2}{n\pi}\,x\,d\cos\frac{n\pi}{2}x$$

$$=-\frac{2}{n\pi}x\cdot\cos\frac{n\pi}{2}x\Big|_0^1+\frac{2}{n\pi}\int_0^1\cos\frac{n\pi}{2}x\,dx-\frac{4}{n\pi}\cos n\pi+\frac{4}{n\pi}\cos\frac{n\pi}{2}$$
$$+\frac{2}{n\pi}x\cos\frac{n\pi}{2}x\Big|_1^2-\frac{2}{n\pi}\int_1^2\cos\frac{n\pi}{2}x\,dx$$

$$=\cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}}+\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}x\Big|_0^1\cancel{-\frac{4}{n\pi}\cos n\pi}\cancel{+\frac{4}{n\pi}\cos\frac{n\pi}{2}}$$
$$\cancel{+\frac{4}{n\pi}\cos n\pi}\cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}}-\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}x\Big|_1^2$$

$$=\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}-\frac{4}{(n\pi)^2}\sin n\pi+\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$=\frac{8}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$=\begin{cases}0 & \text{if } n=2k\\(-1)^{k-1} & \text{if } n=2k-1\end{cases}$$

$$\Rightarrow f(x)=\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{2}$$