Chapter 13

Topic 1

Q1: $w = \frac{1}{z} = \frac{1}{1-i} = \frac{1}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i}{2}$

$\Rightarrow |w| = \frac{\sqrt{2}}{2}$, $\operatorname{Arg} w = \frac{\pi}{4}$

$\Rightarrow w = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Q2:

$\operatorname{Arg} z_1 = \operatorname{Arg}(-2+2i) = \frac{3}{4}\pi$

$\operatorname{Arg} z_2 = \operatorname{Arg}(-6-6i) = -\frac{3}{4}\pi$

$\arg\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \frac{3}{4}\pi - \left(-\frac{3}{4}\pi\right) = \frac{3}{2}\pi$

$\Rightarrow \operatorname{Arg}\frac{z_1}{z_2} = \arg\frac{z_1}{z_2} - 2\pi = \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$

Topic 2

Q3. (a) $\frac{-5}{8}i$

(b) 192

(c) $\frac{\sqrt{17}}{2\sqrt{2}}$

(d) 12

Topic 3

Q4: $z^4 = 1 = \cos 0 + i\sin 0$

$\Rightarrow z = \sqrt[4]{\cos 0 + i\sin 0}$

$= \cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}$ $(k=0,1,2,3)$

$= \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}$

$\Rightarrow z_0 = 1$, $z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$z_2 = \cos\pi + i\sin\pi = -1$, $z_3 = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi = -i$

Q5:

$z^3 = 2-2i = 2\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$

$\Rightarrow z = \left[2\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)\right]^{\frac{1}{3}}$

$= (2\sqrt{2})^{\frac{1}{3}}\left(\cos\frac{-\frac{\pi}{4}+2k\pi}{3} + i\sin\frac{-\frac{\pi}{4}+2k\pi}{3}\right)$ $(k=0,1,2)$

$\Rightarrow z_0 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos(-\frac{\pi}{12}) + i\sin(-\frac{\pi}{12})\right)$

$z_1 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right)$

$z_2 = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$

$= (2\sqrt{2})^{\frac{1}{3}}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

Topic 4

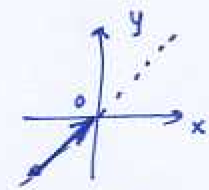
Q6. $\operatorname{Im}(z^2) = \operatorname{Im}\left((x+yi)^2\right) = \operatorname{Im}\left((x^2-y^2)+2xyi\right) = 2xy$

$|z|^2 = x^2+y^2$

$$\Rightarrow f(z) = \begin{cases} \dfrac{2xy}{x^2+y^2} & z=0 \\ 0 & z \neq 0 \end{cases}$$

We'll take the $\lim\limits_{\substack{x\to 0\\ y\to 0}}$ via two different paths.

$$\lim_{\substack{y=x\\ x\to 0}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = 2$$



$$\lim_{\substack{y=-x\\ x\to 0}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{-2x^2}{x^2+(-x)^2} = -2$$

Since $2 \neq -2$,

the $\lim\limits_{z\to 0} \dfrac{2xy}{x^2+y^2}$ doesn't exist.

Q7. $f(x,y) = u + iv$

$u = e^{-x}\cos y \qquad v = -e^{-x}\sin y$

$u_x = -e^{-x}\cos y \qquad v_y = -e^{-x}\cos y$

$u_y = -e^{-x}\sin y \qquad v_x = e^{-x}\sin y$

$$\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

so, the function is analytic

Topic 4

Q8. (a) $f(z) = z + \bar{z}$

$$= (x+iy) + (x-iy) = 2x$$

$\Rightarrow u = 2x.\quad v = 0$

$u_x = 2 \quad v_y = 0 \Rightarrow u_x \neq v_y$. so. $f(z)$ NOT Analytic

(b). $g(z) = 3z - 2\bar{z}$

$$= 3(x+iy) - 2(x-iy) = x + 5yi$$

$u_x = 1 \quad v_y = 5 \quad u_x \neq v_y$. so, $g(z)$ ~~to~~ is not analytic

(c) $h(z) = \dfrac{\bar{z}}{|z|^2} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

$\Rightarrow u = \dfrac{x}{x^2+y^2} \quad v = \dfrac{-y}{x^2+y^2}$

$$u_x = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = \frac{-1\cdot(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{-(-y)\cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$\Rightarrow u_x = v_y \quad , \quad u_y = -v_x$

so, $h(z)$ is analytic.

Q 9 :

Let $z = re^{i\theta}$, then

$$f(z) = \frac{i}{r^2 e^{2i\theta}} = \frac{i}{r^2} e^{-2i\theta} = \frac{i}{r^2}(\cos 2\theta - i \sin 2\theta)$$

$$= \frac{\sin 2\theta}{r^2} + i \frac{\cos 2\theta}{r^2}$$

$\Rightarrow u(r,\theta) = \frac{\sin 2\theta}{r^2}$, $V(r,\theta) = \frac{\cos 2\theta}{r^2}$

$\Rightarrow u_r = -2 \sin(2\theta) \cdot r^{-3}$, $V_\theta = -2\sin(2\theta) \cdot r^{-2}$

$u_\theta = 2 \cos(2\theta) r^{-2}$, $V_r = -2\cos(2\theta) \cdot r^{-3}$

$$\Rightarrow \begin{cases} u_r = \frac{1}{r} V_\theta \\ V_r = -\frac{1}{r} u_\theta \end{cases}$$

$\Rightarrow f(z)$ is analytic when $z \neq 0$

Q 10.

$f(z)$ is analytic $\Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\Rightarrow \begin{cases} u_{xy} = v_{yy} \\ -u_{yx} = v_{xx} \end{cases}$

Since $u_{xy} = u_{yx}$, $v_{xx} + v_{yy} = u_{xy} - u_{yx} = 0$

$\Rightarrow v(x,y)$ is harmonic

Q 11

$\left.\begin{array}{l} u_x = 2x \Rightarrow u_{xx} = 2 \\ u_y = -2y-1 \Rightarrow u_{yy} = -2 \end{array}\right\} \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow u(x,y)$ is harmonic

A harmonic conjugate $v(x,y)$ must satisfy

$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 2x & (1) \\ v_x = 2y+1 & (2) \end{cases}$

$(1) \Rightarrow v(x,y) = 2xy + h(x)$

$\Rightarrow v_x(x,y) = 2y + h'(x) = 2y+1$ (by (2))

Thus, $h'(x) = 1 \Rightarrow h(x) = x + C$

$\Rightarrow v(x,y) = 2xy + x + C$

Q 12

$u_x = 3e^{3x}\cos ay \Rightarrow u_{xx} = 9e^{3x}\cos ay$

$u_y = -ae^{3x}\sin ay \Rightarrow u_{yy} = -a^2 e^{3x}\cos ay$

$u_{xx} + u_{yy} = 0 \Rightarrow (9-a^2)\, e^{3x} \cdot \cos ay = 0$ (u is harmonic)

$\Rightarrow a^2 - 9 = 0$

$\Rightarrow a = \pm 3$

Q 13

$u_x = \cos x \cosh cy \Rightarrow u_{xx} = -\sin x \cosh cy$

$u_y = c \sin x \sinh cy \Rightarrow u_{yy} = c^2 \sin x \cosh cy$

$u_{xx} + u_{yy} = 0 \Rightarrow (c^2-1)\, \sin x \cdot \cosh cy = 0$ (u is harmonic)

$\Rightarrow c^2 - 1 = 0$

$\Rightarrow c = \pm 1$

Topic 5.

Q14. $e^{1/z} = e^{\frac{1}{x+iy}} = e^{\frac{x-iy}{x^2+y^2}} = e^{\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}}$

$$= e^{\frac{x}{x^2+y^2}}\left[\cos\left(\frac{y}{x^2+y^2}\right) - i\sin\left(\frac{y}{x^2+y^2}\right)\right]$$

$Re = e^{\frac{x}{x^2+y^2}} \cos\frac{y}{x^2+y^2}$. $Im = -e^{\frac{x}{x^2+y^2}} \sin\frac{y}{x^2+y^2}$

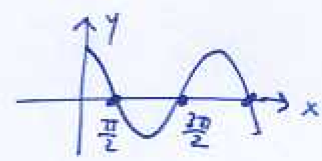
Q15 $e^{-3+\frac{4\pi}{7}i} = e^{-3}\left[\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right]$

$Re = e^{-3}\cos\frac{4\pi}{7}$ $Im = e^{-3}\sin\frac{4\pi}{7}$,

$\left|e^{-3+\frac{4\pi}{7}i}\right| = e^{-3}$

Q16 $\sin(5-2i) = \sin(5)\cosh(2) - i\cos(5)\sinh(2)$

Q17 $\cosh\left(\left(n+\frac{1}{2}\right)\pi i\right) = \cos\left(\left(n+\frac{1}{2}\right)\pi\right) = 0$



Q18 $e^{i3\theta} = (e^{i\theta})^3$

$\Rightarrow (\cos 3\theta + i\sin 3\theta) = (\cos\theta + i\sin\theta)^2$

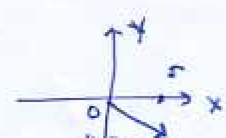
$\qquad\qquad \parallel$

$\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sinh^3\theta$

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

(Real parts of both sides)

Q19 $\text{Ln}(5-4i) = \text{Ln}(5) + i\arctan\left(\frac{-4}{5}\right)$



$\text{Ln}(-2) = \text{Ln}(2) + i\pi$

Q20 $(1+i)^{1-i} = \left[e^{\text{Ln}(1+i)}\right]^{1-i} = e^{(1-i)\left[\text{Ln}\sqrt{2} + \frac{\pi}{4}i\right]} = e^{\left(\text{Ln}\sqrt{2}+\frac{\pi}{4}\right)+\left(\frac{\pi}{4}-\ln\sqrt{2}\right)i}$

Chapter 17

Topic 1.

Q1. (a) TRUE (Associative Law)

(b) TRUE (Distributive Law)

(c) FALSE. $(A+B)^2 = (A+B)(A+B) = AA + BA + AB + BB$

$$= A^2 + BA + AB + BB$$

because there's no commutitive law for matrices $\neq A^2 + 2AB + B^2$

(d) TRUE

Q2. (a) $AB = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 2 & 1 \\ 20 & -10 & 15 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 8 & 2 \\ 6 & 11 \end{bmatrix}$

(c) $A + B$ is not defined

(d) $A - B^T = \begin{bmatrix} -1 & -5 \\ -1 & 2 \\ -1 & 2 \end{bmatrix}$

Topic 2

Q3

Let A be a $m \times n$ matrix and suppose $m > n$

we know that rank A = row rank = column rank

Since column rank $\leq n$, then row rank $\leq n < m$

However, there are m row vectors, thus the row vectors are linearly dependent

The $n > m$ situation is similar.

Q4

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix} \xrightarrow[\substack{(3)-(1)\times 3 \\ (4)-(1)\times 4}]{(2)-(1)\times 2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{vmatrix}$$

$$\xrightarrow[(4)-(2)\times 3]{(3)-(2)\times 2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ the vectors are dependent

Q5

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 0 & 3 \\ 8 & 2 & 3 \\ 5 & 5 & 6 \end{pmatrix}$$ is a 4×3 matrix

$\Rightarrow$ rank $A \leq 3$

$\Rightarrow$ row rank ≤ 3

$\Rightarrow$ row vectors are linearly dependent

$\Rightarrow$ the given vectors are linearly dependent

Topic 3: Null Space.
Q6:

$$A = \begin{pmatrix} 10 & 1 & 2 \\ 0 & 1 & 6 \end{pmatrix}$$

we want $Ax = 0, x = (x_1, x_2, x_3)^T$, which is just

$$\begin{pmatrix} 10 & 1 & 2 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Thus, we have

$$\begin{cases} 10x_1 + x_2 + 2x_3 = 0, & (1) \\ x_2 + 6x_3 = 0, & (2) \end{cases}$$

Look at the second equation, we get

$$x_2 = -6x_3$$

Plug it into the first equation, we get

$$10x_1 - 4x_3 = 0$$

Therefore, x_1 and x_3 are dependent, I could now pick $x_1 = 1$, then $x_3 = 5/2$, again use

$$x_2 = -6x_3$$

we get $x_3 = -15$, thus the null space of A could be spanned by $x = (1, -15, 5/2)^T$, in other words, $x = (1, -15, 5/2)^T$ is a basis of the null space. Notice $rank(A) = 2$, the x has 3 variables, so the dimension of your null space is $3 - 2 = 1$, which coincides with the answer.

Q7:

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 0 & 1 & 6 & 8 \end{pmatrix}$$

we want $Ax = 0, x = (x_1, x_2, x_3, x_4)^T$, which is just

$$\begin{pmatrix} 10 & 1 & 2 & 3 \\ 0 & 1 & 6 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Thus, we have

$$\begin{cases} 10x_1 + x_2 + 2x_3 + 3x_4 = 0, & (1) \\ x_2 + 6x_3 - 8x_4 = 0, & (2) \end{cases}$$

Look at the second equation, we get

$$x_2 = -6x_3 + 8x_4$$

Plug it into the first equation, we get

$$10x_1 - 4x_3 + 11x_4 = 0$$

Now x_1, x_3 and x_4 are dependent, we can choose any values for two of them to determine the third one.

Here, I will choose $x_1 = 1, x_3 = 0$ first, then $x_4 = -10/11$, again use

$$x_2 = -6x_3 + 8x_4$$

we get $x_2 = -80/11$, so $x_{(1)} = (1, -80/11, 0, -10/11)^T$. We are not done yet, remember $rank(A) = 2$, and the x has 4 variables, so the dimension of the null space should be $4 - 2 = 2$, so you still need one more vector. OK, now I choose $x_1 = 0, x_3 = 1$, then $x_4 = 4/11$, again use

$$x_2 = -6x_3 + 8x_4$$

we get $x_2 = -34/11$, so $x_{(2)} = (0, -34/11, 1, 4/11)^T$. Now we are done. The null space of A could be spanned by $x_{(1)}$ and $x_{(2)}$, in other words, $x_{(1)}$ and $x_{(2)}$ is the basis of the A's null space.

Topic 4

Q 8

$$A = \begin{pmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \end{pmatrix} \xrightarrow[(3)-(1)\times\frac{1}{2}]{(2)-(1)\times 2} \begin{pmatrix} 8 & 2 & 5 \\ 0 & 2 & 19 \\ 0 & -1 & -\frac{19}{2} \end{pmatrix}$$

$$\xrightarrow{(3)+(2)\times\frac{1}{2}} \begin{pmatrix} 8 & 2 & 5 \\ 0 & 2 & 19 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank $A = 2$

row basis : $(8 \ \ 2 \ \ 5)$, $(0 \ \ 2 \ \ 19)$

column basis : $(8 \ \ 0 \ \ 0)^T$, $(2 \ \ 2 \ \ 0)^T$

Q 9

$$(A|B) = \left(\begin{array}{ccc|c} -2 & 2 & 6 & 1 \\ 1 & -1 & 2 & 3 \\ -1 & 1 & 3 & 2 \end{array}\right) \xrightarrow[(3)-(1)\times\frac{1}{2}]{(2)+(1)\times\frac{1}{2}} \left(\begin{array}{ccc|c} -2 & 2 & 6 & 1 \\ 0 & 0 & 5 & \frac{7}{2} \\ 0 & 0 & 0 & \frac{3}{2} \end{array}\right)$$

$\Rightarrow$ rank $(A|B) >$ rank (A)

$\Rightarrow$ $Ax = B$ does not have solution

Q 10

False

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } A = \text{rank } B = 2$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A. \quad B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } A^2 = 2 > 1 = \text{rank } B^2$$

Q11

(a)
$$A = \begin{pmatrix} 0 & -6 & 4 \\ 1 & -2 & -2 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{pmatrix}$$

Exchange

$$\xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & -2 & -2 \\ 0 & -6 & 4 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{pmatrix} \xrightarrow[-3\text{Row}1]{-\text{Row}1} \begin{pmatrix} 1 & -2 & -2 \\ 0 & -6 & 4 \\ 0 & -6 & 4 \\ 0 & -6 & 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, the rank of A is 2

(b) Since the rank of A is 2, we know there should be exactly two vectors that form a basis of the column space. This two vectors need to be Linearly independent.

We can check that the first two columns are Linearly independent, so $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ -2 \\ -8 \\ -12 \end{pmatrix}$ can be a basis of the column space.

NOTE the answer is not unique.

Remark: You can also find the basis of the column space by doing Gaussian Column Elimination . .

(c). $(1, -2, -2)$ and $(0, -6, 4)$ form a basis of the row space.

NOTE: Again the answer is not unique!

(d) To solve $AX = 0$, we can use the result by Gaussian Elimination.

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_1 - 2x_2 - 2x_3 = 0$

$\qquad -6x_2 + 4x_3 = 0$

~~if choosing except~~ we

$\Rightarrow \quad x_2 = \frac{2}{3} x_3, \quad x_1 = 2x_2 + 2x_3 = \frac{4}{3} x_3 + 2x_3 = \frac{10}{3} x_3$

$\Rightarrow$ general form of solution is

$$\begin{pmatrix} \frac{10}{3} x_3 \\ \frac{2}{3} x_3 \\ x_3 \end{pmatrix} \quad \text{for any } x_3$$

OR. You can say. the general form of Solution is $c\begin{pmatrix} \frac{10}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$ for any constant c

(e) The dimension of the null space is 1, and the basis of the null space is $\begin{pmatrix} \frac{10}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$

(f). The augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & -6 & 4 & 1 \\ 1 & -2 & -2 & 2 \\ 1 & -8 & 2 & 3 \\ 3 & -12 & -2 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 1 & -8 & 2 & 3 \\ 3 & -12 & -2 & 7 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & -6 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \begin{cases} x_1 - 2x_2 - 2x_3 = 2 \\ \quad -6x_2 + 4x_3 = 1 \end{cases}$$

One particular solution is (by picking $x_3 = 0$

$x_3 = 0$, $x_2 = -\frac{1}{6}$ $x_1 = \frac{5}{3}$

So the general form of solution is

$$c\begin{pmatrix} \frac{10}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{3} \\ -\frac{1}{6} \\ 0 \end{pmatrix} \quad \text{for any } C$$