Math 322. Spring 2008
Review Problems For The Final Exam

Topics for midterm I & II

Topic 1: Complex Numbers
— Specific topics: Polar form of complex number; Operations of complex numbers; Roots of complex number; Continuity, Differentiability and analyticity of complex functions; Cauchy-Riemann equations; harmonic function and harmonic conjugate; Exponential, trigonometric, hyperbolic and logarithmic functions, general power.

Problem 1.1: Let $z_1 = 2 - 2i$, $z_2 = 2 + 3i$, find

$$\text{(a.)} \quad \frac{z_1 + z_2}{\bar{z}_2^2} \qquad \text{(b.)} \quad \mathrm{Im}([(1-i)^8 z_1^2])$$

$$\text{(c.)} \quad \left|\frac{z_1 - z_2}{z_2}\right| \qquad \text{(d.)} \quad \mathrm{Re}((z_1 + 1) z_2)$$

Problem 1.2: Find all the solutions for $z^3 = 1$.

Problem 1.3: Find out whether the following function is continuous at $z = 0$.

$$f(z) = \begin{cases} \frac{\mathrm{Im}(z)}{1-|z|}, & z \neq 0; \\ 0, & z = 0. \end{cases}$$

Problem 1.4: Use Cauchy-Riemann equations to check whether the following function is analytic.

$$f(z) = 2z - 2\bar{z}$$

Problem 1.5: Verify that $u(x, y) = x^3 - 3xy^2 - 2x$ is harmonic in the whole complex plane, find a harmonic conjugate function $v(x, y)$ of $u(x, y)$, and find $f(z) = u(x, y) + iv(x, y)$.

Problem 1.6: Compute $\sin(2 + 3i)$, e^{-3+2i}, $\cosh(3 + n\pi i)$, $\mathrm{Ln}(-3 + i)$. Write your answer in the form $a + bi$.

Topic 2: Linear Algebra

— Specific topics: Matrix Operations; Linear Independence of vectors; Linear system of equations; Rank, row space, column space, basis; Determinant, Inverse; Eigenvalues and eigenvectors.

Problem 2.1: Let

$$A=\begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 5 & 0 \end{bmatrix}, B=\begin{bmatrix} 2 & 1 & 3 \\ 5 & -1 & 2 \end{bmatrix}$$

Calculate the following products or sums or give reasons why they are not defined.

(a) AB (b) BA (c) $A+B$ (d) $A-B^T$

Problem 2.2: Are the following vectors linearly independent? [3 4 7], [2 0 3], [0 2 1]?

Problem 2.3: Consider the following matrix

$$A=\begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

(a.) Let $x = [x_1, x_2, x_3]^T$. Find the general form of solutions for the homogeneous linear system of equations $AX = 0$, and find the dimension of the null space of A.

(b.) Let $x = [x_1, x_2, x_3, x_4]^T$. Find the general form of solutions for the homogeneous linear system of equations $A^T X = 0$, and find the dimension of the null space of A^T.

(c.) Let $b = [1, 1, 0, 1]^T$. Does the nonhomogeneous system of equations $Ax = b$ have solution(s)? If your answer is yes, find the general form of the solution(s).

(d.) Find a basis of the column space of A.

(e.) Find a basis of the row space of A.

(f.) Find the rank of A.

Problem 2.4: Consider the following matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & -1 \end{pmatrix}.$$

(a). Find the determinant of matrix A.

(b). Find the inverse of matrix A, or argue why it doesn't exist.

Problem 2.5: Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix}$$

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Topic 3: ODEs

— Linear independency of functions, Linear ODE/ ODE system (Existence and uniqueness of solutions, homogeneous, inhomogeneous; Sturm-Liouville equations; Radius of convergence of power series, solving ODE with method of power series

Problem 3.1: **Problem 1**: Find an ODE for which the given functions $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are solutions. Verify these two functions are independent.

Problem 3.2: Solve the following **inhomogeneous** initial value problem.

$$\begin{aligned} y''' + 3y'' + 3y' + y = 30e^{-x} \\ y(0) = 3 \\ y'(0) = -3 \\ y''(0) = -47 \end{aligned}$$

Problem 3.3: Let $A = \begin{pmatrix} 5 & -28 & -18 \\ -1 & 5 & 3 \\ 3 & -16 & -10 \end{pmatrix}$ and $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Find the general form of solution to the following system of equations.

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

Problem 3.4: Solve the following Sturm-Liouville problem, where $\lambda > 0$

$$y'' + 4y' + (\lambda + 4)y = 0, \ y(0) = 0, \ y'(1) = 0$$

Problem 3.5: Find the radius of convergence of the following series:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{3^m} x^m.$$

Problem 3.6: Solve the following ODE by power series.

$$y'' = y + x^2$$

Topics after Midterm II

Topic 4: Fourier Series

Problem 4.1: $f(x)$ has period 2π, and

$$f(x) = |x|, \ \ -\pi \leq x \leq -\pi$$

Find the Fourier series of $f(x)$.

Problem 4.2: $f(x)$ has period π, and

$$f(x) = x^2, \ \ 0 \leq x \leq \pi$$

Find the Fourier series of $f(x)$.

Problem 4.3:

$$f(x) = \frac{\pi - x}{4}, \ \ 0 \leq x \leq \pi$$

Find the sine series of $f(x)$.

Topic 5: Fourier Transform

Problem 5.1: Find the Fourier transform of the following function.

$$f(x) = e^{-2x^2}, \qquad \text{for} - \infty < x < \infty$$

Problem 5.2: Find the Fourier transform of the following function.

$$f(x) = \begin{cases} 1, & \text{for} |x| \leq 1; \\ 0, & \text{for} |x| > 1. \end{cases}$$

Problem 5.3: Circle **ALL** the statement that is true about the Fourier transforms.

(a). $\mathcal{F}(af(x) + bg(x)) = a(f(x)) + b\mathcal{F}(g(x))$, where a and b are constants.

(b). $\mathcal{F}^{-1}(\hat{f}(k)\hat{g}(k)) = \frac{1}{\sqrt{2\pi}} f(x) * g(x)$, where $*$ is the convolution operator.

(c). $\mathcal{F}(\frac{df(x)}{dx}) = ik\mathcal{F}(f(x))$.

(d). $\mathcal{F}(\frac{d^2 f(x)}{dx^2}) = -k^2\mathcal{F}(f(x))$.

Topic 6: Partial Differential Equation (PDE) on a finite domain

Problem 6.1: Solve the boundary value problem defined by Laplace's equation using the method of separation of variables:

$$u_{xx} + u_{yy} = 0 \text{ on the square } 0 < x, y < 10$$

subject to the boundary conditions

$$u(0, y) = 0,\ u(10, y) = 0,\ u(x, 10) = 0, u(x, 0) = 100\sin(\pi x/10).$$

Problem 6.2: Solve the initial value problem defined by heat equation using the method of separation of variables:

$$u_t = u_{xx}\ 0 \le x \le \pi,\ t \ge 0$$

subject to the boundary conditions

$$u(0, t) = 0,\ u(\pi, t) = 0$$

and the initial condition

$$u(x, 0) = 1,\ 0 \le x \le \pi$$

Problem 6.3: Solve the initial value problem defined by heat equation using the method of separation of variables:

$$u_t = u_{xx}\ 0 \le x \le 5,\ t \ge 0$$

subject to the boundary conditions

$$u(0, t) = 0,\ u(5, t) = 0$$

and the initial condition

$$u_t(x, 0) = x,\ 0 \le x \le 5$$

Topic 7: Partial Differential Equation (PDE) on infinite domain

Problem 7.1: Use Fourier transform to solve the following heat equation on $(-\infty, \infty)$, where E is a constant.

$$\begin{aligned} &u_t = E^2 u_{xx} \\ &u(x, 0) = e^{-2x^2}, \qquad \text{for} - \infty < x < \infty \end{aligned}$$

Topic 8: Laplace transform and properties

Problem 8.1: Find the Laplace transform for the following function.

$$f(x) = \begin{cases} \cos(x), & \text{for } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 8.2: Find the Laplace transform for the following function.

$$f(x) = \begin{cases} 1, & \text{for } x > 3; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 8.3: Circle **ALL** the statement that is true about the Laplace transforms.

(a). $\mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$, where a and b are constants.

(b). $\mathcal{L}(\int_1^\infty f(x)) = \frac{1}{k}\mathcal{L}(f(x))$

(c). $\mathcal{L}(\frac{df(x)}{dx}) = k\mathcal{L}(f(x))$.

(d). $\mathcal{L}(\frac{d^2f(x)}{dx^2}) = k^2\mathcal{L}(f(x))$.

(e). If $\mathcal{L}(f(x)) = F(s)$, then $\mathcal{L}(e^{ax}f(x)) = F(s-a)$

Problem 8.4: Use Laplace transform to solve the following initial value problem.

$$\begin{aligned} \frac{d^2y}{dt^2} + y &= t \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Problem 8.5: Use Laplace transform to solve the following initial value problem.

$$\begin{aligned} \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y &= 0 \\ y(0) &= 3 \\ y'(0) &= 0 \end{aligned}$$