Topic 1:

Question 1

(a) $\frac{-32+43i}{169}$

(b) -128

(c) $\frac{5}{\sqrt{13}}$

(d) 12

Question 2

$z^3=1=\cos 0+i\sin 0$

$\Rightarrow z=\sqrt[3]{\cos 0+i\sin 0}$

$=\cos\frac{0+2k\pi}{3}+i\sin\frac{0+2k\pi}{3}$

$=\cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}$ $(k=0,1,2)$

$\Rightarrow z_0=1$ $\quad z_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$z_2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Q3: When $z \to 0$, then $|z| \to 0$

thus $1-|z| \to 1$

also if $z \to 0$, $Im(z) \to 0$

thus $z \to 0$

$$\Rightarrow \frac{Im(z)}{1-|z|} \to 0$$

Since $f(0)=0$, then $f(z)$ is continuous at $z=0$

Note Title 5/5/2008

#1.4

$f(z) = 2z - 2\bar{z}$

$= 2(x+iy) - 2(x-iy)$

$= 0 + 4iy$

$\Rightarrow u = 0, \quad v = 4y$

$u_x = 0 \qquad v_y = 4$

$\Rightarrow u_x \neq v_y \quad \Rightarrow$ $f(z)$ is not analytic!

Note Title 5/4/2008

#1.5

$u(x,t)=x^3-3xy^2-2x$

To check it is harmonic, we need to check $u_{xx}+u_{yy}=0$

$u_x = 3x^2 - 3y^2 - 2$, $u_{xx} = 6x$

$u_y = -6xy$, $u_{yy} = -6x$

$\Rightarrow u_{xx} + u_{yy} = 6x + (-6x) = 0$

$\Rightarrow u(x,t)$ is harmonic!

- To find its harmonic conjugate $v(x,y)$, we need $v(x,y)$ to satisfy the Cauchy-Riemann eqns

$$\begin{cases} u_x = v_y & \Rightarrow v_y = u_x = 3x^2 - 3y^2 - 2 \quad (1) \\ v_x = -u_y & \Rightarrow v_x = -u_y = -(-6xy) = 6xy \quad (2) \end{cases}$$

Integrate (1) w.r.t. y:

$$\int v_y\,dy = \int (3x^2 - 3y^2 - 2)\,dy \Rightarrow v = 3x^2y - y^3 - 2y + h(x) \quad (3)$$

Integrate (2) w.r.t x:

$$\int v_x\,dx = \int 6xy\,dx \Rightarrow v = 3x^2y + g(y) \quad (4)$$

$$-y^3 - 2y = g(y), \quad h(x) = 0$$

In order to make (3) and (4) equal, we must take

$g(y) = -y^3 - 2y$, $h(x) = 0$,

So, $\boxed{v = 3x^2y - y^3 - 2y}$

Finally, $f(x,y) = u + iv$
$= (x^3 - 3xy^2 - 2x) + i(3x^2y - y^3 - 2y)$
$= (x + iy)^3 - 2(x+iy) = \boxed{z^3 - 2z}$

$\underline{\# 1.6}$

$$\sin(2+3i) = \frac{e^{(2+3i)i} - e^{-(2+3i)i}}{2i}$$

$$= \frac{e^{-3+2i} - e^{3-2i}}{2i}$$

$$= \frac{e^{-3}(\cos 2 + i\sin 2) - e^{3}(\cos 2 - i\sin 2)}{2i}$$

$$= \frac{(e^{-3}\cos 2 - e^{3}\cos 2) + i(e^{-3}\sin 2 + e^{3}\sin 2)}{2i}$$

$$= \left(\frac{e^{-3}\sin 2 + e^{3}\sin 2}{2}\right) - \left(\frac{e^{-3}\cos 2 - e^{3}\cos 2}{2}\right) i$$

$$= \left(\frac{e^{-3}+e^{3}}{2}\sin 2\right) + \left(\frac{e^{3}-e^{-3}}{2}\cos 2\right) i$$

$$= \underbrace{\cosh(3)\sin(2)}_{a} + \underbrace{\sinh(3)\cos(2)}_{b}\, \underset{i}{i}$$

Note: For this problem, you can also use the formula provided by the book (Page 627)

$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

but it is good to know how this formula is worked out, as shown in this solution.

$$e^{-3+2i} = e^{-3}(\cos 2 + i\sin 2)$$

$$= \underbrace{e^{-3}\cos 2}_{a} + \underbrace{i e^{-3}\sin 2}_{ib}$$

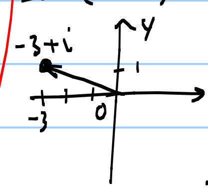
$$\cosh(3+n\pi i) = \cosh(3)\cos(n\pi) + i\sinh(3)\sin(n\pi)$$

$$= \underbrace{\cosh(3)\cdot(-1)^n}_{a} + \underbrace{0\, i}_{+bi}$$

Here, we used the formula provided on textbook page 629, but you could also get the same result directly using definition

$$\cosh(z) = \frac{e^{z}+e^{-z}}{2}$$

$$\operatorname{Ln}(-3+i) = \operatorname{Ln}\left(\sqrt{3^2+1^2}\right) + i\operatorname{Arg}(-3+i)$$



$$\operatorname{Arg}(-3+i) = \arctan\left(\frac{1}{-3}\right) + \pi$$

so, $\operatorname{Ln}(-3+i) = \underbrace{\operatorname{Ln}(\sqrt{10})}_{a} + \underbrace{i\left(\arctan\left(\frac{1}{-3}\right)+\pi\right)}_{+ib}$

Topic 2

Q1:

(a) $AB=\begin{pmatrix} 8 & -3 & 1 \\ 5 & -1 & 2 \\ 10 & 5 & 15 \end{pmatrix}$

(b) $BA=\begin{pmatrix} 13 & 5 \\ 5 & 9 \end{pmatrix}$

(c) $A+B$ is not defined

(d) $A-B^T=\begin{pmatrix} -3 & -3 \\ -1 & 2 \\ 2 & -2 \end{pmatrix}$

Q2:

$$\begin{pmatrix} 3 & 4 & 7 \\ 2 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{(2)-(1)\times\frac{2}{3}} \begin{pmatrix} 3 & 4 & 7 \\ 0 & -\frac{8}{3} & -\frac{5}{3} \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{(3)+(2)\times\frac{3}{4}} \begin{pmatrix} 3 & 4 & 7 \\ 0 & -\frac{8}{3} & -\frac{5}{3} \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}$$

$\Rightarrow$ rank $=3$ $\Rightarrow$ vectors independent

2.3

(a) To find the null space of A, we need to solve
$$Ax = 0$$

$$\begin{pmatrix} \boxed{1} & -2 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 3 & 0 \end{pmatrix} \xrightarrow[\text{Row 4 - Row 1}]{\text{Row 2 - Row 1, Row 3 - Row 1}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 2 & 2 \\ 0 & 5 & 0 \end{pmatrix}$$

Done !

$$\xrightarrow[\text{Row 4 - 5 Row 2}]{\text{Row 3 - 2 Row 2}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{0} \\ 0 & 0 & -5 \end{pmatrix} \xrightarrow{\text{Exchange Row 3 \& 4}} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Done

Done with Gaussian Elimination

$$\Rightarrow \begin{cases} x_1 - 2x_2 = 0 & \longrightarrow x_1 = 0 \\ x_2 + x_3 = 0 & \longrightarrow x_2 = 0 \\ -5x_3 = 0 & \longrightarrow x_3 = 0 \end{cases}$$

(Solve starting from the last equation)

So, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the only solution to $Ax = 0$,

Or, you can say the dimension of the null space is <u>zero</u>! (Since there is no such a non-zero vector to even form a basis)

(b) To find the null space of A^T, we need to solve $A^T x = 0$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{Row 2} + 2\,\text{Row 1}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{Row 3} - \text{Row 2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow{\text{Row 4} - \text{Row 3}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 & \longrightarrow x_1 = x_3 \\ x_2 + 2x_3 + 5x_4 = 0 & \longrightarrow x_2 = -2x_3 \\ -5x_4 = 0 & \rightarrow x_4 = 0 \end{cases}$$

So, the general solution to $A^T x = 0$ is $\begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix}$.

Since there is one free variable x_3 in the general solution, we say the null space of A^T has dimension 1.

(c) $$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \xrightarrow[\text{Row 2} - \text{Row 1}]{\text{Row 4} - \text{Row 1},\ \text{Row 3} - \text{Row 1}} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & -1 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\text{Row 4} - 5\,\text{Row 2}]{\text{Row 3} - 2\,\text{Row 2}} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -5 & 0 \end{array}\right) \Longrightarrow 0x_1 + 0x_2 + 0x_3 = -1$$

i.e. $0 = -1$

Apparently, the equation doesn't have any solution.

So, with $b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $Ax = b$ doesn't have a solution.

(d). it is equivalent to ask to find a basis for the space spanned by vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\-1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}.$$

We need to put them in rows $\begin{pmatrix}1 & 1 & 1 & 1\\ -2 & -1 & 0 & 3\\ 0 & 1 & 2 & 0\end{pmatrix} = A^T$ and do row operations.

To save some effort, note this is just A^T, and we have done Gaussian Elimination on A^T before.

$$\begin{pmatrix}1 & 1 & 1 & 1\\ -2 & -1 & 0 & 3\\ 0 & 1 & 2 & 0\end{pmatrix} \xrightarrow[\text{(see part (b))}]{} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 5\\ 0 & 0 & 0 & -5\end{pmatrix} \quad \text{basis}$$

So, $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\5\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-5\end{pmatrix}$ form a basis of the column space.

(e) $$\begin{pmatrix}1 & -2 & 0\\ 1 & -1 & 1\\ 1 & 0 & 2\\ 1 & 3 & 0\end{pmatrix} \xrightarrow[\text{(see part (a))}]{} \begin{pmatrix}1 & -2 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & -5\end{pmatrix} \quad \text{basis}$$

So, $(1,-2,0)$, $(0,1,1)$, $(0,0,-5)$ form a basis of the row space.

(f). From part (d) & (e), we see although column space and row space are two totally different thing, when it comes to the number of vectors in the basis, they share a number 3.

Thus. the rank of the matrix is 3

#2.4. (a)

$$\det(A) = \overbrace{(-1)\cdot \det\begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} + 0 + 0}^{\text{Row 1 expansion}}$$

$$= (-1)\cdot\left((-2) - 0\right) = (-1)(-2) = 2$$

(b) Since $\det(A) = 2 \neq 0$, A^{-1} exist.

$$A^{-1} = \frac{\begin{pmatrix} +(-2) & -(-2) & +2 \\ -0 & +1 & -(-2) \\ +0 & -0 & +(-2) \end{pmatrix}^T}{\det(A)} = \frac{\begin{pmatrix} -2 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix}^T}{2}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

#2.5

$$\det\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 1 \\ 0 & -4 & -3-\lambda \end{pmatrix} = \underbrace{(1-\lambda)\det\begin{pmatrix} 2-\lambda & 1 \\ -4 & -3-\lambda \end{pmatrix}}_{\text{column 1 expansion}} + 0 + 0$$

$$= (1-\lambda)\left[(2-\lambda)(-3-\lambda) + 4\right]$$

$$= (1-\lambda)(\lambda^2 + \lambda - 2) = (1-\lambda)(\lambda+2)(\lambda-1) = 0$$

$$\underbrace{\lambda = 1, \quad \lambda = 1,}_{\text{double roots}} \qquad \underbrace{\lambda = -2}_{\text{simple root}}$$

So, the matrix has two eigen values,

double

$\lambda = 1$ (with algebraic multiplicity 2)

simple

$\lambda = -2$ (with algebraic multiplicity 1)

Find eigenvectors:

For $\lambda = 1$

$$\begin{pmatrix} 1-1 & 1 & 2 \\ 0 & 2-1 & 1 \\ 0 & -4 & -3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \longrightarrow x_2 = -x_3 \\ -4x_2 - 4x_3 = 0 \end{cases}$$

plug in

plug in

$$\Rightarrow x_3 = 0, \quad x_2 = 0, \quad x_1 = \text{Anything}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \text{ especially } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is an eigenvector for } \lambda = 1$$

Note, although $\lambda = 1$ is a double root, it only has one linearly independent eigenvector, thus we say the geometric multiplicity of $\lambda = 1$ is 1

For $\lambda = -2$,

$$\begin{pmatrix} 1+2 & 1 & 2 \\ 0 & 2+2 & 1 \\ 0 & -4 & -3+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3x_1 + x_2 + 2x_3 = 0 \\ 4x_2 + x_3 = 0 \\ -4x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{7}{12} x_3 \\ x_2 = -\frac{1}{4} x_3 \\ x_3 = x_3 \end{cases}$$

$$\Rightarrow \begin{pmatrix} -\frac{7}{12} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \text{ is an eigenvector for } \lambda = -2$$

So, $\lambda = -2$ also has geometric multiplicity 1

Topic 3

Q 1: $\left.\begin{array}{l} y_1 = e^{-2x} \\ y_2 = xe^{-2x} \end{array}\right\} \Rightarrow -2$ is a double root of the characteristic function

$\Rightarrow$ The characteristic equation is

$(\lambda+2)^2 = \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow$ The ODE is $y'' + 4y' + 4y = 0$

$$W(e^{-2x}, xe^{-2x}) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix}$$

$$= (1-2x)e^{-4x} + 2xe^{-4x}$$

$$= e^{-4x} \neq 0$$

$\Rightarrow$ y_1 and y_2 are independent

Q 2 $\lambda^3+3\lambda^2+3\lambda+1=(\lambda+1)^3=0$

$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

$y_p = C x^3 e^{-x}$

$y_p' = C(3x^2 - x^3)e^{-x}$

$y_p'' = C(6x - 6x^2 + x^3)e^{-x}$

$y_p''' = C(6 - 18x + 9x^2 - x^3)e^{-x}$

$\Rightarrow C(6-18x+9x^2-x^3)+3C(6x-6x^2+x^3)+3C(3x^2-x^3)$
$\quad + Cx^3 = 30$

$\Rightarrow 6C = 30$

$\Rightarrow \quad C = 5 \Rightarrow y_p = 5x^3 e^{-x}$

$\Rightarrow y = y_h + y_p = (c_1 + c_2 x + c_3 x^2)e^{-x} + 5x^3 e^{-x}$

$y(0) = c_1 = 3$

$y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}$

$y'(0) = -3 + c_2 = -3 \Rightarrow c_2 = 0$

$y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}$

$y''(0) = 3 + 2c_3 = -47 \Rightarrow c_3 = -25$

$\Rightarrow y = (3 - 25x^2)e^{-x} + 5x^3 e^{-x}$

Q3.

$$\det(A-\lambda E)$$

$$= \det\begin{pmatrix} 5-\lambda & -28 & -18 \\ -1 & 5-\lambda & 3 \\ 3 & -16 & -10-\lambda \end{pmatrix}$$

$$\doteq 3\lambda(1-\lambda^2) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -1$$

For $\lambda_1 = 0$. $\quad v_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$. $\quad V_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\lambda_3 = -1$ $\quad V_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow y = c_1\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}e^{x} + c_3\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}e^{-x}$$

#3.4: $y'' + 4y' + (\lambda+4)y = 0$ $(\lambda > 0)$

characteristic equation: $r^2 + 4r + (\lambda+4) = 0$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(\lambda+4)}}{2} = \frac{-4 \pm \sqrt{-4\lambda}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-\lambda}}{2} \overset{\uparrow}{=} \frac{-4 \pm 2\sqrt{\lambda}\,i}{2} = -2 \pm \sqrt{\lambda}\,i$$

because $\lambda > 0 \Rightarrow -\lambda < 0 \Rightarrow \sqrt{-\lambda} = \sqrt{\lambda}\,i$

$\Rightarrow$ general Solution: $y = Ae^{-2x}\cos(\sqrt{\lambda}x) + Be^{-2x}\sin(\sqrt{\lambda}x)$

To determine A and B, use the boundary conds.

$y(0) = A + B\cdot 1 \cdot 0 = A + 0 = A = 0$

$\Rightarrow y = Be^{-2x}\sin(\sqrt{\lambda}x)$

$\Rightarrow y'(x) = -2Be^{-2x}\sin(\sqrt{\lambda}x) + \sqrt{\lambda}Be^{-2x}\cos(\sqrt{\lambda}x)$

$$y'(1) = \underbrace{-2Be^{-2x}\sin(\sqrt{\lambda}) + \sqrt{\lambda}Be^{-2x}\cos(\sqrt{\lambda})}_{\parallel} = 0$$

$$\left(-2\sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda})\right)\cancel{Be^{-2x}} = 0$$

$$\Rightarrow \tan(\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{2}$$

so, the solution is

$y = Be^{-2x}\sin(\sqrt{\lambda}x)$, where B is an arbitrary constant, and λ is such that it must satsify $\tan(\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{2}$ (although in this case, the exact value for λ is hard to get!)

3.5

a_m shouldn't include x

$$a_m = \frac{(-1)^m}{3^m} \qquad a_{m+1} = \frac{(-1)^{m+1}}{3^{m+1}}$$

$$k = \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to\infty} \left| \frac{(-1)^{m+1}}{3^{m+1}} \cdot \frac{3^m}{(-1)^m} \right|$$

$$= \lim_{m\to\infty} \left| \frac{1}{3} \right| = \frac{1}{3}$$

always non-negative because of "absolute value"

$$R = \frac{1}{k} = 3$$

#3.6 $y'' = y + x^2$

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

Then $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$

$y'' = y + x^2 \Rightarrow$

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) = (a_0 + a_1 x + a_2 x^2 + \cdots) + x^2$$

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \cdots) = a_0 + a_1 x + (a_2 + 1) x^2 + a_3 x^3 + \cdots$$

By comparison,

$$\begin{cases} 2a_2 = a_0 \\ 6a_3 = a_1 \\ 12a_4 = a_2 + 1 \\ 20a_5 = a_3 \\ \vdots \end{cases} \Rightarrow$$

We notice there are two free variables a_0 and a_1 in the solution, and every other terms depend on a_0, a_1 only

a_0, a_1, $a_2 = \frac{a_0}{2}$, $a_3 = \frac{a_1}{6}$, $a_4 = \frac{a_2+1}{12} = \frac{\frac{a_0}{2}+1}{12} = \frac{a_0+2}{24}$,

Free Free

So, $y(x) = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_0+2}{24} x^4 + \cdots$

Remark: You might want to ask: how would I know how many free variables I should have in the solution? The answer lies in the order of the equation itself. Since this is a 2nd order ODE (because of y''), we expect to have two free variables a_0, a_1.

Topic 4

Q 1 :

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1) = \begin{cases} -\frac{4}{\pi}\frac{1}{n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$b_n = 0 \quad (|x| \text{ is even})$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Q2:

$$2L=\pi \Rightarrow L=\frac{\pi}{2}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}x^2dx=\frac{1}{3}\pi^2$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos 2nx\,dx$$

$$=-\frac{2}{n\pi}\int_0^{\pi}x\sin 2nx\,dx$$

$$=\frac{1}{n^2\pi}x\cos 2nx\Big|_0^{\pi}-\frac{1}{n^2\pi}\int_0^{\pi}\cos 2nx\,dx$$

$$=\frac{1}{n^2}$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}x^2\sin 2nx\,dx$$

$$=-\frac{1}{n\pi}x^2\cos 2nx\Big|_0^{\pi}+\frac{2}{n\pi}\int_0^{\pi}x\cos 2nx\,dx$$

$$=-\frac{\pi}{n}-\frac{1}{n^2\pi}\int_0^{\pi}\sin 2nx\,dx$$

$$=-\frac{\pi}{n}$$

$$\Rightarrow f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{1}{k^2}\cos 2nx-\frac{\pi}{n}\sin 2nx\right)$$

Q3

Do the odd extension:

$L = \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi - x}{4}\sin nx\,dx$$

$$= \frac{1}{2n}$$

$$\Rightarrow \frac{\pi - x}{4} = \sum_{n=1}^{\infty} \frac{1}{2n}\sin nx$$

#5.1 Lets denote the Fourier transform of $f(x)$ as $F(k)$

then $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2x^2} e^{-ikx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{ikx}{2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{ikx}{2} - \left(\frac{k}{4}\right)^2\right) - \frac{k^2}{8}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x + \frac{ik}{4}\right)^2 - \frac{k^2}{8}} dx$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{\sqrt{a}}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right) \cdot e^{-\frac{k^2}{8}}$$

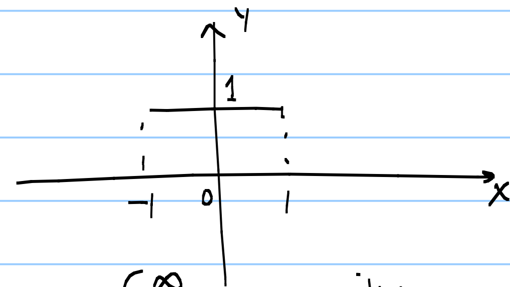
$$= \frac{1}{2} e^{-\frac{k^2}{8}}$$

Remark. $\int_{-\infty}^{\infty} e^{-2\left(x + \frac{ik}{4}\right)^2} dx$

$= \int_{-\infty}^{\infty} e^{-2x^2} dx = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)$

As if $\frac{ik}{4}$ was not there.

#5.2



$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} e^{-ikx} \Big|_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} \left(e^{-ik} - e^{ik} \right) = \frac{1}{\sqrt{2\pi} \cdot k} \frac{e^{ik} - e^{-ik}}{i}$$

$$= \frac{2}{\sqrt{2\pi} \cdot k} \cdot \frac{e^{ik} - e^{-ik}}{2i} = \frac{2}{\sqrt{2\pi}\, k} \sin(k)$$

#5.3 multiple choice. The answer is (a, b, c, d)

Topic 6.

Q1. $u(x,y)=F(x)G(y)$

$$\frac{F''(x)}{F(x)}=-\frac{G''(y)}{G(y)}=-K$$

$$\Rightarrow \begin{cases} F''(x)+KF(x)=0 & (1) \\ G''(y)-KG(y)=0 & (2) \end{cases}$$

$(1) \Rightarrow F(x)=A\cos(\sqrt{K}x)+B\sin(\sqrt{K}x)$

Apply $u(0,y)=u(10,y)=0 \Rightarrow A=0$ & $K=(\frac{n\pi}{10})^2$

$\Rightarrow F_n(x)=\sin\frac{n\pi x}{10}$, $K=(\frac{n\pi}{10})^2$, $n=1,2,\cdots$

$(2) \Rightarrow G''(y)-(\frac{n\pi}{10})^2 G(y)=0$

$\Rightarrow G_n(y)=A_n e^{(\frac{n\pi}{10})y}+B_n e^{-(\frac{n\pi}{10})y}$

$u(x,10)=0 \Rightarrow G_n(10)=0 \Rightarrow A_n e^{n\pi}+B_n e^{-n\pi}=0$

$\Rightarrow G_n(y)=B_n[e^{(-\frac{n\pi}{10})y}-e^{[(\frac{n\pi}{10})y-2n\pi]}]$

$\Rightarrow u_n(x,y)=F_n(x)G_n(y)=\sin\frac{n\pi x}{10}[e^{(-\frac{n\pi}{10})y}-e^{[(\frac{n\pi}{10})y-2n\pi]}]$

$u(x,y)=\sum_n A_n u_n(x,y)$

$u(x,0)=\sum_n \underbrace{A_n(1-e^{-2n\pi})}_{B_n}\sin(\frac{n\pi x}{10})=100\sin(\frac{\pi x}{10})$

$\Rightarrow B_n=100$ and $n=1$

$\Rightarrow u(x,y)=\frac{100}{1-e^{-2\pi}}\sin(\frac{\pi x}{10})[e^{(-\frac{\pi}{10}y)}-e^{[(\frac{\pi}{10}y)-2\pi]}]$

Q 2: $u(x,t) = F(x)G(t)$

$u_t = u_{xx} \Rightarrow \frac{\dot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)} = -K$

$$\begin{cases} F''(x) + KF(x) = 0 & (1) \\ G'(t) + KG(t) = 0 & (2) \end{cases}$$

$(1) \Rightarrow F(x) = A\cos(\sqrt{K}x) + B\sin(\sqrt{K}x)$

Apply boundary conditions, $\Rightarrow A = 0$ & $K = n^2$

$\Rightarrow F_n(x) = \sin(nx)$ & $K = n^2$

$(2) \Rightarrow G'(t) = -n^2 G(t)$

$\Rightarrow G_n(t) = e^{-n^2 t}$

$\Rightarrow u_n(x,t) = F_n(x) G_n(t) = \sin(nx) e^{-n^2 t}$

$u(x,t) = \sum_n A_n u_n(x,t)$

$u(x,0) = \sum_n A_n \sin(nx) = 1$

$$\Rightarrow A_n = \frac{2}{\pi}\int_0^\pi \sin(nx) \cdot u(x,0)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin(nx)\,dx$$

$$= \frac{2}{\pi}\cdot\left[-\frac{1}{n}\cos(nx)\right]\Big|_0^\pi$$

$$= \frac{4}{n\pi}$$

$$\Rightarrow u(x,t) = \sum_n \frac{4}{n\pi} \sin nx \; e^{-n^2 t}$$

Q3: $u(x,t) = F(x)G(t)$

$$u_t = u_{xx} \Rightarrow \frac{\dot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)} = -K$$

$$\begin{cases} F''(x) + KF(x) = 0 & (1) \\ G'(t) + KG(t) = 0 & (2) \end{cases}$$

$(1) \Rightarrow F(x) = A\cos(\sqrt{K}x) + B\sin(\sqrt{K}x)$

Apply Boundary conditions $\Rightarrow A = 0$ & $K = (\frac{n\pi}{5})^2$

$\Rightarrow F_n(x) = \sin\frac{n\pi x}{5}$ & $K = (\frac{n\pi}{5})^2$

$(2) \Rightarrow G'(t) = -(\frac{n\pi}{5})^2 G(t)$

$\Rightarrow G_n(t) = e^{-(\frac{n\pi}{5})^2 t}$

$\Rightarrow u_n(x,t) = F_n(x) \cdot G_n(t) = \sin\frac{n\pi x}{5} \cdot e^{-(\frac{n\pi}{5})^2 t}$

$u(x,t) = \sum_n A_n u_n(x,t) = \sum_n A_n \sin\frac{n\pi x}{5} e^{-(\frac{n\pi}{5})^2 t}$

$u_t(x,t) = \sum_n -(\frac{n\pi}{5})^2 A_n \sin\frac{n\pi x}{5} e^{-(\frac{n\pi}{5})^2 t}$

$$u_t(x,0) = \sum_n \underbrace{-(\frac{n\pi}{5})^2 A_n}_{B_n} \sin\frac{n\pi x}{5} = x$$

$$\Rightarrow B_n = \frac{2}{5}\int_0^5 x \cdot \sin\frac{n\pi x}{5}\,dx \qquad (L=5)$$

$$= \frac{2}{5}\left[-\frac{5}{n\pi}x\cos\frac{n\pi x}{5}\Big|_0^5 + \int_0^5 \frac{5}{n\pi}\cos\frac{n\pi x}{5}dx\right]$$

$$= -\frac{10}{n\pi}\cos n\pi$$

$\Rightarrow A_n = B_n \cdot \left[-(\frac{5}{n\pi})^2\right] = \frac{250}{(n\pi)^3} \cdot \cos n\pi \Rightarrow u(x,t) = \sum_n \frac{250}{(n\pi)^3}\cos n\pi \cdot \sin\frac{n\pi x}{5} e^{-(\frac{n\pi}{5})^2 t}$

#7.1 $u_t = E^2 u_{xx}$, $u(x,0) = e^{-2x^2}$

Take the Fourier transform on both sides $\boxed{\text{w.r.t } x}$

$$F[u_t] = F[E^2 u_{xx}] \quad (*)$$

Since $F[u_{xx}] = (ik)^2 F[u] = -k^2 F[u]$

$$F[u_t] = (F[u])_t$$

For simplicity of notation, lets call $F[u] = \hat{f}(k)$

so, the equation $(*)$ becomes

$$\frac{\partial \hat{f}(k)}{\partial t} = -E^2 k^2 \hat{f}(k).$$

Solve for $\hat{f}(k)$ to get $\Longrightarrow$ $\hat{f}(k) = Ce^{-E^2 k^2 t}$ $(**)$

To determine constant C, lets take the F.T. on the initial condition, and we get

$$\hat{f}(k)\Big|_{t=0} = \frac{1}{2} e^{-\frac{k^2}{8}}$$ (see problem 5.1)

If we plug $t=0$ into $(**)$, we will get

$$\hat{f}(k)\Big|_{t=0} = C$$

so, $C = \frac{1}{2} e^{-\frac{k^2}{8}}$

Thus $\hat{f}(k) = \frac{1}{2} e^{-\frac{k^2}{8}} e^{-E^2 k t}$ is fully determined

Remember, $\hat{f}(k)$ is the Fourier Transform of the solution $u(x,t)$. To find $u(x,t)$, we need to take the inverse F.T.,

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{2} e^{-\frac{k^2}{8}} e^{-E^2 k^2 t}}_{\hat{f}(k)} e^{ikx} \, dx$$

#8.1

$$L[f](s) = \int_0^\infty e^{-st} \cos(t)\, dt$$

$$= -\frac{1}{s} e^{-st} \cdot \cos(t)\Big|_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st}(-\sin(t))\, dt$$

$$= \frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-st} \sin(t)\, dt$$

$$= \frac{1}{s} - \frac{1}{s}\left(-\frac{1}{s} e^{-st} \sin(t)\Big|_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st} \cos(t)\, dt\right)$$

$$= \frac{1}{s} - \frac{1}{s}\left(0 + \frac{1}{s}\int_0^\infty e^{-st} \cos(t)\, dt\right)$$

$$= \frac{1}{s} - \frac{1}{s^2}\int_0^\infty e^{-st} \cos(t)\, dt$$

Let $\int_0^\infty e^{-st} \cos(t)\, dt = I$, then the equation above becomes

$$I = \frac{1}{s} - \frac{1}{s^2} I$$

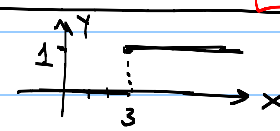
solve for I: $\Longrightarrow$

$$\left(1 + \frac{1}{s^2}\right) I = \frac{1}{s} \Rightarrow I = \frac{\frac{1}{s}}{1 + \frac{1}{s^2}} = \frac{s}{s^2+1}$$

So, the Laplace transform $L[f](s) = \int_0^\infty e^{-st} \cos t\, dt = \boxed{\dfrac{s}{s^2+1}}$

#8.2

$$f(x) = \begin{cases} 1 & x \geq 3 \\ 0 & x < 3 \end{cases}$$



$$L[f](s) = \int_0^\infty e^{-sx} f(x)\, dx = \int_3^\infty e^{-st} \cdot 1\, dx$$

$$= -\frac{1}{s} e^{-st}\Big|_3^\infty = \frac{1}{s} e^{-3s}$$

Note that, since the fcn shifted to the right by 3 units, compared with $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, its L-transform is then multiplied by e^{-3s}.

#8.3

Multiple choice. the answer is $\underline{(a, e)}$

Corrections for the wrong choices

0

(b). $\mathfrak{L}(\int^{\infty} f(x)) = \frac{1}{k}\mathfrak{L}(f(x))$

(c). $\mathfrak{L}(\frac{df(x)}{dx}) = k\mathfrak{L}(f(x)).- f(0)$

(d). $\mathfrak{L}(\frac{d^2 f(x)}{dx^2}) = k^2\mathfrak{L}(f(x)).- sf(0) - f'(0)$

#8.4 $y'' + y = x, \qquad y(0) = 0, \quad y'(0) = 0$

Take the Laplace transform on both sides.

For simplicity, denote $L[y](s)$ as $F(s)$

So, $L[y](s) = F(s)$

Then $L[y''] = s^2 F(s) - s\overset{0}{y(0)} - \overset{0}{y'(0)}$

$= s^2 F(s) -$

And $L[x] = \frac{1}{s^2}$ (By using table)

So, the equation becomes: $s^2 F(s) + F(s) = \frac{1}{s^2}$

Solve for $F(s)$: $F(s) = \dfrac{\frac{1}{s^2}}{s^2+1} = \dfrac{1}{s^2(s^2+1)}$

The inverse Laplace transform can be obtained in 3 ways

① Directly lookup in the table (Fortunately, it is there ...)

Table: $\boxed{\dfrac{1}{s^2(s^2+w^2)} \longleftrightarrow \dfrac{1}{w^3}(wt - \sin wt)}$, so, let $w=1$, we get the inverse L-transform as $(t - \sin t)$.

If it is not in the table, you can use one of the following two ways:

② From the table, you know the following relation

$f(t) \longrightarrow F(s)$

$\sin(t) \longrightarrow \dfrac{1}{s^2+1}$

$-\cos(t) + 1 = \int_0^t \sin(t)\,dt \longrightarrow \dfrac{1}{s(s^2+1)}$

$\boxed{t - \sin(t)} = \int_0^t (-\cos t + 1)\,dt \longrightarrow \dfrac{1}{s^2(s^2+1)}$

Same thing!

$\vdots$

③ Use "partial fractions" to decompose $\underbrace{\dfrac{1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^2+1}}$

Question: Do you know how I got this form?

Then by simplifying

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2+1} = \frac{A[s(s^2+1)] + B(s^2+1) + Cs^2}{s^2(s^2+1)}$$

$$= \frac{As^3 + (B+C)s^2 + As + B}{s^2(s^2+1)}$$

$\Rightarrow A = 0, \quad B + C = 0, \quad A = 0, \quad B = 1$

$\Rightarrow A = 0, \quad B = 1, \quad C = -1$

So, $\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} + \frac{-1}{(s^2+1)}$

$\frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} t \qquad \frac{-1}{(s^2+1)} \xrightarrow{\mathcal{L}^{-1}} -\sin t$

So, $\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \boxed{t - \sin t}$

Again, the same!

Thus, no matter which of the above three ways you choose in order to find the inverse Laplace transform of $\frac{1}{s^2(s^2+1)}$, you will get $t - \sin(t)$ eventually.

In conclusion, the solution to the equation is $y(t) = t - \sin t$

8.5 $\quad y'' - y' - 2y = 0, \quad y(0)=3, \quad y'(0)=0$

Take the Laplace transform on both sides, and let $L[y](s) = F(s)$.

Then $L[y''] = s^2 F(s) - s\underset{3}{y(0)} - \underset{0}{y'(0)}$

$$= s^2 F(s) - 3s$$

$$L[y'] = sF(s) - \underset{3}{y(0)} = sF(s) - 3$$

$$L[2y] = 2F(s) \quad (\text{Table})$$

$$\Rightarrow \left(s^2 F(s) - 3s\right) - \left(sF(s) - 3\right) - 2F(s) = 0$$

$$\Rightarrow (s^2 - s - 2)F(s) + (3 - 3s) = 0$$

$$\Rightarrow F(s) = \frac{3s-3}{s^2 - s - 2} = \frac{3s-3}{(s+1)(s-2)}$$

Partial Fractions: $\dfrac{3s-3}{(s+1)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s-2}$

$$= \frac{A(s-2) + B(s+1)}{(s+1)(s-2)} = \frac{(A+B)s + (B-2A)}{(s+1)(s-2)}.$$

$$\Rightarrow \begin{cases} A + B = 3 \\ B - 2A = 3 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 1 \end{cases}$$

$$\Rightarrow F(s) = \frac{2}{s+1} + \frac{1}{s-2}$$

$$\downarrow L^{-1} \qquad \downarrow L^{-1}$$

$$2e^{-t} \qquad 1e^{2t}$$

(Look up in table)

$$\Rightarrow y(t) = 2e^{-t} + e^{2t}$$