1.(20pts) FOR THIS ENTIRE PROBLEM, LET $z = 1 - i$.

a. (10pts) Evaluate z^3 in the Cartesian form $x + iy$ AND in the polar form $re^{i\theta}$ using the principal argument.

$$z = \sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$z^3 = \left(\sqrt{2}\, e^{-i\frac{\pi}{4}}\right)^3 = 2^{\frac{3}{2}} e^{-i\frac{3\pi}{4}}$$ (polar form)

$$z^3 = 2^{\frac{3}{2}}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= 2^{\frac{3}{2}}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -2 - 2i$$ (Cartesian form)

b. (10pts) Evaluate $\operatorname{Ln} z$ and $\operatorname{Ln}(z^3)$ in the form $x + iy$.

$$\operatorname{Ln}(z^3) = \operatorname{Ln}(2\sqrt{2}) - i\frac{3\pi}{4}$$

2. (25pts)

a. (15 pts) Find all solutions to $z^3 = -8i$.

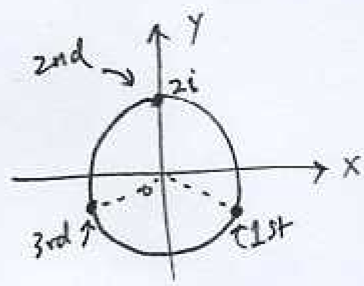
$$z^3 = 8e^{-i\frac{\pi}{2}}$$

$$z = \left[8 e^{i\left(-\frac{\pi}{2}+2k\pi\right)}\right]^{1/3} \qquad k = 0, 1, 2$$

1st. $z = 2e^{i\left(-\frac{\pi}{6}\right)} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i$

2nd $z = 2e^{i\left(\frac{\pi}{2}\right)} = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2i$

3rd $z = 2e^{i\left(\frac{7\pi}{6}\right)} = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = -\sqrt{3} - i$



b. (10 pts) Find the modulus and principal argument of the number $\exp\left(\ln 3 + \frac{28}{3}\pi i\right)$.

$$\left| e^{\ln 3 + \frac{28}{3}\pi i} \right| = e^{\ln 3} = 3$$

$$\mathrm{Arg}\left(e^{\ln 3 + \frac{28}{3}\pi i}\right) = \frac{28}{3}\pi - \underbrace{5(2\pi)}_{\text{must be an integer multiple of } 2\pi} = \frac{28}{3}\pi - 10\pi = -$$

3. (25pts)

a. (10pts) Show that $u(x,y) = e^x \sin y$ is a harmonic function.

$$u_x = e^x \sin y \qquad u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y \qquad u_{yy} = -e^x \sin y$$

$$\Rightarrow u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

$$\Rightarrow u \text{ is harmonic}$$

b. (15pts) Find real numbers a and b such that the function $f(z) = f(x+iy) = 3x^2 + ay^2 + i(bxy)$ is an analytic function.

$$u = 3x^2 + ay^2 \qquad v = bxy$$

$$u_x = 6x \qquad v_x = by$$

$$u_y = 2ay \qquad v_y = bx$$

$$\begin{cases} u_x = v_y \Rightarrow 6x = bx \Rightarrow b = 6 \\ u_y = -v_x \Rightarrow 2ay = -by \Rightarrow a = \frac{-b}{2} \Rightarrow a = -\frac{6}{2} \end{cases}$$

c. (Extra credit: 5pts) Write $f(z)$ as a function of z alone.

$$f(z) = 3x^2 - 3y^2 + 6xyi = 3(x+iy)^2 = 3z^2$$

4. (10pts)
Show that

$$\cos^2 z + \sin^2 z = 1$$

for any complex number z. (Hint: express $\cos z$ and $\sin z$ in terms of complex exponentials.)

by definition.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\Rightarrow \cos^2 z = \frac{(e^{iz} + e^{-iz})^2}{4}$$

$$= \frac{e^{2iz} + 2 + e^{-2iz}}{4}$$

$$\sin^2 z = \frac{(e^{iz} - e^{-iz})^2}{-4}$$

$$= \frac{e^{2iz} - 2 + e^{-2iz}}{-4}$$

$$\Rightarrow \cos^2 z + \sin^2 z = \frac{(e^{2iz} + 2 + e^{-2iz}) - (e^{2iz} - 2 + e^{-2iz})}{4}$$

$$= \frac{4}{4} = 1$$

5. (25pts) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

a. (10pts) Find a basis for the row space of A.

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix} \xrightarrow[(3)-(1)]{(2)-(1)\times 2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -4 & -5 \\ 0 & -4 & -5 \end{pmatrix} \xrightarrow{(3)-(2).} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

So, $(1, 1, 3)$ and $(0, -4, -5)$ form a basis of the row space.

b. (5pts) What is the rank of A?

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c. (10pts) Does the system $AX = B$, with $B = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$, have a solution? If so, how many solutions are there?

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & -2 & 1 & 0 \\ 1 & -3 & -2 & -3 \end{array}\right) \xrightarrow[(3)-(1)]{(2)-(1)\times 2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -4 & -5 & -6 \\ 0 & -4 & -5 & -6 \end{array}\right)$$

$$\xrightarrow{(3)-(2)} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -4 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -4x_2 - 5x_3 &= -6 \end{aligned}$$

x_3 is a free parameter. For any x_3, we can solve for x_2, and x_1 afterwards.

6. (20pts)

a. (10pts) Find a basis for nullspace of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{(2)-(1)} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 5 \end{pmatrix} \Leftrightarrow \begin{matrix} x_1 + x_2 - x_3 = 0 \\ x_2 + 5x_3 = 0 \end{matrix}$$

Since x_3 is the only free parameter, we choose. $x_3 = 1$.

then $x_2 = -5$, $x_1 = x_3 - x_2 = 1 - (-5) = 6$

so, the basis for the null space is $\begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix}$

b. (10pts) Is the following set of vectors linearly independent? Justify your answer.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -5 \\ -2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -4 & 2 & -5 & -2 \end{bmatrix} \xrightarrow[(4)+(1)\times 4]{(2)+(1)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{(3)-(2)} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the vectors given are linear dependent

7. (30 pts) Let

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 3 & 0 \end{pmatrix}.$$

a.(12pts) Calculate the following or say "not defined": $A+B$, $A+B^T$, AB, B^TA.

$$A + B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 6 & 6 \end{pmatrix}, \quad A + B^T \text{ NOT defined}$$

NOT EQUAL

AB = NOT defined. because A is 3×2, B is 3×2.

$$B^T A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 18 \\ 3 & -4 \end{pmatrix}$$

b. (8pts) Suppose **A is a 4 x 10 matrix with real entries.**

Which of the following could be the dimensions of the row space $R(A)$, column space $C(A)$, and nullspace $N(A)$ (You do not have to justify your answer):

a. $\dim(C(A)) = 4$, $\dim(R(A)) = 10$, $\dim(N(A)) = 0$

b. $\dim(C(A)) = 3$, $\dim(R(A)) = 3$, $\dim(N(A)) = 7$

c. $\dim(C(A)) = 3$, $\dim(R(A)) = 4$, $\dim(N(A)) = 7$

d. $\dim(C(A)) = 4$, $\dim(R(A)) = 3$, $\dim(N(A)) = 7$

e. $\dim(C(A)) = 4$, $\dim(R(A)) = 4$, $\dim(N(A)) = 0$

c. (10pts) Give an example of a set of three different vectors in $\mathbb{R}^3$ whose span is two-dimensional. You do not have to justify your answer.

$(1, 0, 0)$ $(0, 1, 0)$, $(1, 1, 0)$