Final Exam Problems

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ 3 & 7 & -2 \end{pmatrix}$$

a. Find all solutions to the equation $Ax = 0$.
b. Find all solutions (if any) to $Ax = b$ where

$$b = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

c. Find all solutions (if any) to $Ax = b$ where

$$b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

2.

a. Find all the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$.

b. Find the general solution of the homogeneous ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

3.

a. Write the following differential equation as a first order system:

$$\frac{d^3y}{dx^3} + x^2\frac{dy}{dx} - y^3 = 0.$$

Express your answer as $Y_0' = \cdots$, $Y_1' = \cdots$, etc.

b. For the following, circle ALL statements about the above differential equation which are necessarily true:

(i) The equation is homogeneous.
(ii) The equation is linear.

4.

a. Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 1 & \text{if } -1 < x \leq 0 \\ -2 & \text{if } 0 < x \leq 1 \end{cases}$$

extended to have period 2. It may help to know that $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$ for any integer n.

b. At which points in the interval $-1 \le x \le 1$ does the series not converge to the value of the function? To which values does the series converge at these points?

5.

a. Consider the following partial differential equation for $u(x, y)$:

$$\begin{aligned} u_{xx} + u_{yy} - 3u_y &= 0, \\ u(x,0) &= 0, \\ u(x,1) &= 0 \\ u(0,y) &= 0 \\ u(2,y) &= \sin(4\pi y). \end{aligned}$$

Use separation of variables to obtain two ODE associated with this PDE. (DO NOT solve them.)

b. One (and only one) of those ODE's can be formed into a Sturm-Liouville equation with homogeneous boundary conditions using the boundary conditions from the PDE. State which one and give the boundary conditions.

6.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for $u(x, t)$ on a finite domain $0 \le x \le 10$, with boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(10,t) &= 0 \end{aligned}$$

Recall that the general solution of the wave equation with these boundary conditions is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{10} + B_n \sin \frac{n\pi ct}{10} \right) \sin \frac{n\pi x}{10}.$$

If the equation is given initial conditions

$$\begin{aligned} u(x,0) &= \sin(5\pi x), \\ \frac{\partial u}{\partial t}(x,0) &= \sin(\pi x), \end{aligned}$$

then find the particular solution for $u(x, t)$ (that is, find the coefficients A_n and B_n). Note: you may leave your answer with terms like $A_n = \sin \frac{3\pi n}{10} + \cos \pi n$ without further simplifying.

7.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

on the WHOLE LINE ($x \in (-\infty, \infty)$). Find $\hat{u}(w,t)$, the Fourier transform of the solution $u(x,t)$. DO NOT solve for $u(x,t)$ (only its Fourier tranform). Note: the answer should contain $\hat{f}(w)$.

8. For the following Sturm-Liouville problem, find all POSITIVE eigenvalues λ together with corresponding eigenfunctions.

$$y'' = -\lambda y$$
$$y'(0) = 0, \;\; y'(2) = 0$$

9. Consider the heat equation

$$u_t = c^2 u_{xx}.$$

We want to solve this equation for $0 < x < 1$ and for all t with boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

(**Notice the derivative in the boundary conditions**) and with the initial condition

$$u(x,0) = x.$$

a. Separate the variables $u(x,t) = F(x)G(t)$ and find 2 ordinary differential equations satisfied by F and G.

b. Using the boundary conditions, find the F_n.

c. Find the G_n and write down the eigenfunctions u_n.

d. Write down the general solution $u(x,t)$ and use the initial condition to find the coefficients.

e. What is $\lim_{t \to \infty} u(x,t)$