Math 322. Spring 2015
Review Problems for Midterm 1

Chapter 13 (Complex Numbers):

Topic 1: Polar form of complex number .

Question 1.

Let $z = 1 - i$. Evaluate $w = 1/z$ in polar form, with the principal argument.

Answer:

$$z = 1 - i = \sqrt{2}e^{-i\pi/4}$$

so

$$\frac{1}{z} = \frac{1}{\sqrt{2}}e^{i\pi/4}.$$

Question 2.

Let $z_1 = -2 + 2i$ and $z_2 = -6 - 6i$. Evaluate $Arg(z_1/z_2)$.

Answer:

$$z_1 = 2\sqrt{2}e^{3\pi i/4}$$
$$z_2 = 6\sqrt{2}e^{-3\pi i/4}$$

so

$$\frac{z_1}{z_2} = \frac{1}{3}e^{3\pi i/2} = \frac{1}{3}e^{-\pi i/2}$$

so $Arg\,(z_1/z_2) = -\pi/2$.

Topic 2: Operations of complex numbers.

Question 3.

Let $z_1 = 3 + 2i$, $z_2 = 2 - 2i$, find

$$\text{(a)} \quad \frac{z_1 + z_2}{\bar{z}_2^2} \qquad \text{(b)} \quad \text{Im}([(1-i)^8 z_1^2])$$

$$\text{(c)} \quad \left|\frac{z_1 - z_2}{z_2}\right| \qquad \text{(d)} \quad \text{Re}((z_1 + 1)z_2)$$

Answer: a) $\frac{z_1+z_2}{\bar{z}_2^2} = \frac{5}{(2+2i)^2} = -\frac{5}{8}i.$

b) $\text{Im}((1-i)^8 z_1^2) = \text{Im}\left((-2i)^4 (3+2i)^2\right) = 192.$

c) $\left|\frac{z_1 - z_2}{z_2}\right| = \left|\frac{1+2i}{2-2i}\right| = \frac{\sqrt{5}}{2\sqrt{2}}.$

d) $\text{Re}\left((z_1+1)\,z_2\right) = \text{Re}\left((4+2i)(2-2i)\right) = 8 + 4 = 12.$

Topic 3: Roots of complex number .

Question 4.

Find all the solutions for $z^4 = 1$.

Answer: $z = 1, -1, i, -i$.

Question 5.

Find all the solutions for $z^3 = 2 - 2i$.

Answer: $z = 8^{1/6}e^{-i\pi/12}, 8^{1/6}e^{7i\pi/12}, 8^{1/6}e^{-i9\pi/12}$

Topic 6: Exponential, trigonometric, hyperbolic and logarithmic functions, general power.

Question 6.

Let $z = x + iy$. Find the Re and Im of $e^{1/z}$.

Answer:

$$e^{1/z} = e^{\bar{z}/|z|^2} = \frac{e^x}{(x^2+y^2)}\left(\cos\frac{-y}{(x^2+y^2)} + i\sin\frac{-y}{(x^2+y^2)}\right)$$

so $\operatorname{Re} e^{1/z} = \frac{e^x}{(x^2+y^2)}\cos\frac{y}{(x^2+y^2)}$ and $\operatorname{Im} e^{1/z} = -\frac{e^x}{(x^2+y^2)}\sin\frac{-y}{(x^2+y^2)}$.

Question 7.

Find the Re, Im and modulus of $e^{-3+\frac{4\pi}{7}i}$.

Answer: $\operatorname{Re}\left(e^{-3+\frac{4\pi}{7}i}\right) = e^{-3}\cos\frac{4\pi}{7}$, $\operatorname{Im}\left(e^{-3+\frac{4\pi}{7}i}\right) = e^{-3}\sin\frac{4\pi}{7}$, $\left|e^{-3+\frac{4\pi}{7}i}\right| = e^{-3..}$

Question 8.

Compute $\sin(5 - 2i)$.

Answer:

$$\begin{aligned}\sin(5-2i) &= \frac{e^{i(5-2i)} - e^{-i(5-2i)}}{2i} \\ &= \frac{e^{2+5i} - e^{-2-5i}}{2i} \\ &= \frac{e^2(\cos 5 + i\sin 5) - e^{-2}(\cos 5 - i\sin 5)}{2i} \\ &= \cosh 2\sin 5 - i\sinh 2\cos 5.\end{aligned}$$

Question 9.

Compute $\cosh((n+\frac{1}{2})\pi i)$, where n is an integer.

Answer:

$$\begin{aligned}\cosh((n+\frac{1}{2})\pi i) &= \frac{e^{((n+\frac{1}{2})\pi i)} + e^{-((n+\frac{1}{2})\pi i)}}{2} \\ &= \frac{1}{2}(-i+i) = 0.\end{aligned}$$

Another way to see this is that

$$\cosh iz = \cos z.$$

Question 10.

Show the following identity is true. (Hint: You may need to use the identity $e^{inx} = (e^{ix})^n$).

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

Answer:

$$e^{i3\theta} = \cos 3\theta + i \sin\theta$$

but also

$$e^{i3\theta} = \left(e^{i\theta}\right)^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right).$$

Now just compare the real parts.

Question 11.

Compute $\mathrm{Ln}(5-4i)$, $\mathrm{Ln}(-2)$.

Answer: These are easier if we write in polar form. Notice that

$$5 - 4i = \sqrt{41}e^{-i\arctan\frac{4}{5}}$$

and so

$$Ln\,(5-4i) = \ln\sqrt{41} - i\arctan\frac{4}{5}$$

and

$$Ln\,(-2) = \ln 2 + i\pi.$$

Question 12.

Find the principal value of $(1+i)^{1-i}$.

Answer: Recall that the principal value is

$$\begin{aligned}(1+i)^{1-i} &= e^{(1-i)Ln(1+i)} \\ &= e^{(1-i)\left(\ln\sqrt{2}+i\pi/4\right)} \\ &= e^{\ln\sqrt{2}+\pi/4+i\left(\frac{\pi}{4}-\ln\sqrt{2}\right)}\end{aligned}$$

and so we get

$$\sqrt{2}e^{\pi/4}\left(\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right).$$

Chapter 7 (Linear Algebra):

Topic 1: Matrix Operations.

Question 1.

Which of the following equations may not be true? Why not?
(a) $A(BC) = (AB)C$
(b) $(A + B)C = AC + BC$
(c) $(A + B)^2 = A^2 + 2AB + B^2$
(d) $(AB)^T = B^T A^T$

Answer: a is true by associativity of matrix multiplication. b is true by distributivity. c is false becasue the middle term should be $AB + BA$ which is not equal to $2AB$ in general. d is true since it is a property of transpose.

Question 2.

Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -2 & 3 \end{bmatrix}$$

Calculate the following products or sums or give reasons why they are not defined.
(a) AB (b) BA (c) $A + B$ (d) $A - B^T$

Answer:

$$AB = \begin{bmatrix} 2 & 6 & -1 \\ 3 & 2 & 1 \\ 20 & -10 & 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 8 & 2 \\ 6 & 11 \end{bmatrix}$$

$$A - B^T = \begin{bmatrix} -1 & -5 \\ -1 & 2 \\ -1 & 2 \end{bmatrix}$$

and $A + B$ does not make sense since the dimensions of the matrices are not the same.

Topic 2: Linear system of equations, row operations

Question 3.

Let

$$A = \begin{pmatrix} -2 & 2 & 6 \\ 1 & -1 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Does the system $Ax = B$ with $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ admit solutions? If so, how many? Find them.

Answer: No solutions. We have that the augmented matrix satisfies:

$$\begin{bmatrix} -2 & 2 & 6 & 1 \\ 1 & -1 & 2 & 3 \\ -1 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & 6 & 1 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 0 & 1.5 \end{bmatrix}$$

so there are no solutions.

Question 4.

Let

$$A = \begin{bmatrix} 0 & -6 & 4 \\ 1 & -2 & -2 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{bmatrix}$$

Let $b = [1, 2, 3, 7]^T$. Does the following system of equations have solution(s)? If your answer is yes, find the general form of the solution(s).

$$Ax = b.$$

Answer:

The augmented matrix satisfies:

$$\begin{bmatrix} 0 & -6 & 4 & 1 \\ 1 & -2 & -2 & 2 \\ 1 & -8 & 2 & 3 \\ 3 & -12 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 1 & -8 & 2 & 3 \\ 3 & -12 & -2 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & -6 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -2 & 2 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + 2t + 2\left(-\frac{1}{6} + \frac{2}{3}t\right) \\ -\frac{1}{6} + \frac{2}{3}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{10}{3}t + \frac{5}{3} \\ \frac{2}{3}t - \frac{1}{6} \\ t \end{bmatrix}.$$