Math 322. Spring 2015
Review Problems for Midterm 2

Linear Algebra:

Topic: Linear Independence of vectors .

Question 1.

Explain why if A is not square, then either the row vectors or the column vectors of A are linearly dependent.

Question 2.

Are the following vectors linearly independent: [1 2 3 4], [2 3 4 5], [3 4 5 6], [4 5 6 7]?

Question 3.

Are the following vectors linearly independent: [3 4 7], [2 0 3], [8 2 3], [5 5 6]?

Topic: Linear system of equations, rank, row space, column space, basis

Question 4.

Find the rank, a basis for the row space, and a basis for the column space of the following matrix.

$$\begin{pmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \\ \end{pmatrix}$$

Question 5.

Let

$$A = \begin{pmatrix} -2 & 2 & 6 \\ 1 & -1 & 2 \\ -1 & 1 & 3 \\ \end{pmatrix}.$$

Does the system $Ax = B$ with $B = \begin{pmatrix} 1 \\ 3 \\ 2 \\ \end{pmatrix}$ admit solutions? If so, how many?

Question 6.

Does rank A = rank B imply rank A^2 = rank B^2? If yes, justify your answer; otherwise, give a counterexample.

Question 7.

Let

$$A = \begin{bmatrix} 0 & -6 & 4 \\ 1 & -2 & -2 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{bmatrix}$$

(a) Find the rank of A.

(b) Find a basis of the column space of A.

(c) Find a basis of the row space of A.

(d) Let $x = [x_1, x_2, x_3]^T$. Find the general form of solutions for the homogeneous linear system of equations $AX = 0$.

(e) Find the dimension of the null space of A.

(f) Let $b = [1, 2, 3, 7]^T$. Does the following system of equations have solution(s)? If your answer is yes, find the general form of the solution(s).

$$AX = b.$$

Topic: Determinant.

Question 8: Find the determinant of the following matrix.

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}.$$

Question 9: Let

$$B = \begin{bmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{bmatrix}$$

Find $\det(B)$.

Topic: Eigenvalues and eigenvectors.

Question 10:

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Question 11:

Find the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Understanding questions:

Question 12: Is it possible for a linear system of equations to have exactly 10 solutions? Why or why not?

Question 13: Is it possible for a linear system of equations to have no solution at all? If so, give an example. If not, explain why.

Question 14: Give an example of a 3 by 3 matrix whose rank is 1. What is the dimension of the null space of the matrix you just found? Explain.

Question 15: Give an example of three 3-dimensional vectors that do not span $\mathbb{R}^3$. Choose the vectors so that no two vectors are proportional to one another.

Question 16: Give an example of three 3-dimensional vectors with non-zero entries that span $\mathbb{R}^3$.

Question 17: For a system of n equations with n unknowns of the form $Ay = b$, list four ways to tell if there is a unique solution or not.

Ordinary differential equations:

Topic: Linear independency of functions)

Question 1:

Find an ODE for which the given functions $y_1 = \cos x$ and $y_2 = \sin x$ are solutions. Verify these two functions are linearly independent.

Question 2:

Find an ODE for which the given functions $y_1 = e^x$ and $y_2 = xe^x$ are solutions. Verify these two functions are linearly independent.

Topic: Linear ODE (Existence and uniqueness of solutions, solving homogeneous linear ODE)

Question 3: Find the general form of solution to the following equation.

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$$

Question 4: Consider the following initial value problem.

$$\begin{aligned} \frac{d^2y}{dx^2} - 4y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

(a). Does this initial value problem have a solution? Is the solution unique?

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

Topic: Linear ODE system

Question 5: Consider the following initial value problem.

$$\frac{dy_1}{dt} = y_1 + 2y_2$$
$$\frac{dy_2}{dt} = 5y_1 - 2y_2$$
$$y_1(0) = 2$$
$$y_2(0) = 9.$$

(a). Does this initial value problem have a solution? Is the solution unique?

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

Question 6: Let

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 5 & -4 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Find the general form of solution to the following system of equations.

$$\frac{dY}{dt} = AY.$$

Question 7: Consider again the ODE given in Question 3.

(a). Convert this problem into an first-order ODE system.

(b). Solve this ODE system, and compare the solution to the solution you found for Question 3.

Understanding questions:

Question 8: What can you say about the set of solutions to a homogeneous linear differential equation?

Question 9: Why do we look at the Wronskian to decide whether a set of functions is linearly independent?

Question 10: If the Wronskian of a set of functions is equal to zero for a particular value of x, does it necessarily mean that the functions are linearly dependent? Why or why not?

Question 11: What condition should the Wronskian of n functions satisfy for these functions to be linearly independent?

Question 12: If you are given a basis of solutions to a linear differential equation, do you know how to write down the general solution?