Math 322. Spring 2015
Review Problems for Midterm 2

Linear Algebra:

Topic: Linear Independence of vectors .

Question 1.

Explain why if A is not square, then either the row vectors or the column vectors of A are linearly dependent.

Answer (one). There have to be either more rows or more columns. Since the dimension of the row space equals the dimension of the column space, that dimension cannot be more than the smaller of the number of columns and number of rows.

Question 2.

Are the following vectors linearly independent: [1 2 3 4], [2 3 4 5], [3 4 5 6], [4 5 6 7]?

Answer: Linearly dependent. Notice that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} = 0$$

Question 3.

Are the following vectors linearly independent: [3 4 7], [2 0 3], [8 2 3], [5 5 6]?

Answer: Linearly dependent. There are never more than 3 linearly independent vectors in $\mathbb{R}^3$.

Topic: Linear system of equations, rank, row space, column space, basis

Question 4.

Find the rank, a basis for the row space, and a basis for the column space of the following matrix.

$$\begin{pmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \end{pmatrix}$$

Answer: For the row space, we do row operations

$$\begin{pmatrix} 8 & 2 & 5 \\ 16 & 6 & 29 \\ 4 & 0 & -7 \end{pmatrix} \sim \begin{pmatrix} 8 & 2 & 5 \\ 0 & 2 & 19 \\ 0 & -1 & -\frac{19}{2} \end{pmatrix} \sim \begin{pmatrix} 8 & 2 & 5 \\ 0 & 2 & 19 \\ 0 & 0 & 0 \end{pmatrix}$$

so the row space is spanned by

$$\begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 19 \end{bmatrix}.$$

Similarly, we can look at

$$\begin{pmatrix} 8 & 16 & 4 \\ 2 & 6 & 0 \\ 5 & 29 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & 0 \\ 8 & 16 & 4 \\ 5 & 29 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & 0 \\ 0 & -8 & 4 \\ 0 & 14 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

so the column space is spanned by

$$\begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ 4 \end{bmatrix}$$

Question 5.

Let

$$A = \begin{pmatrix} -2 & 2 & 6 \\ 1 & -1 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Does the system $Ax = B$ with $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ admit solutions? If so, how many?

Answer: Since

$$\begin{aligned} \begin{pmatrix} -2 & 2 & 6 & 1 \\ 1 & -1 & 2 & 3 \\ -1 & 1 & 3 & 4 \end{pmatrix} &\sim \begin{pmatrix} 1 & -2 & 2 & 3 \\ -2 & 2 & 6 & 1 \\ -1 & 1 & 3 & 4 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & -2 & 10 & 7 \\ 0 & -1 & 5 & 7 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & -2 & 10 & 7 \\ 0 & 0 & 0 & \frac{7}{2} \end{pmatrix} \end{aligned}$$

there are no solutions.

Question 6.

Does rank A = rank B imply rank A^2 = rank B^2? If yes, justify your answer; otherwise, give a counterexample.

Answer: No. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question 7.

Let

$$A = \begin{bmatrix} 0 & -6 & 4 \\ 1 & -2 & -2 \\ 1 & -8 & 2 \\ 3 & -12 & -2 \end{bmatrix}$$

(a) Find the rank of A.

(b) Find a basis of the column space of A.

(c) Find a basis of the row space of A.

(d) Let $x = [x_1, x_2, x_3]^T$. Find the general form of solutions for the homogeneous linear system of equations $AX = 0$.

(e) Find the dimension of the null space of A.

(f) Let $b = [1, 2, 3, 7]^T$. Does the following system of equations have solution(s)? If your answer is yes, find the general form of the solution(s).

$$AX = b.$$

Topic: Determinant.

Question 8: Find the determinant of the following matrix.

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}.$$

Answer: -5.

Question 9: Let

$$B = \begin{bmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{bmatrix}$$

Find $\det(B)$.

Answer: $(d - c)\,(c + d)\,(b - a)\,(a + b)$

Topic: Eigenvalues and eigenvectors.

Question 10:

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The eigenvalues are -3, $3-2\sqrt{2}$, and $3+2\sqrt{2}$ with eigenvectors $\begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2\sqrt{2}+1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2\sqrt{2}+1 \\ -2 \\ 1 \end{bmatrix}$ respectively.

Question 11:

Find the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Answer: The eigenvalues are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ with corresponding eigenvectors $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Understanding questions:

Question 12: Is it possible for a linear system of equations to have exactly 10 solutions? Why or why not?

Question 13: Is it possible for a linear system of equations to have no solution at all? If so, give an example. If not, explain why.

Question 14: Give an example of a 3 by 3 matrix whose rank is 1. What is the dimension of the null space of the matrix you just found? Explain.

Question 15: Give an example of three 3-dimensional vectors that do not span $\mathbb{R}^3$. Choose the vectors so that no two vectors are proportional to one another.

Question 16: Give an example of three 3-dimensional vectors with non-zero entries that span $\mathbb{R}^3$.

Question 17: For a system of n equations with n unknowns of the form $Ay = b$, list four ways to tell if there is a unique solution or not.

Ordinary differential equations:

Topic: Linear independency of functions)

Question 1:

Find an ODE for which the given functions $y_1 = \cos x$ and $y_2 = \sin x$ are solutions. Verify these two functions are linearly independent.

Answer:

$$y'' = -y.$$

We can use the Wronskian to figure out linear dependence:

$$\det \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = 1 \neq 0$$

so they are linearly independent.

Question 2:

Find an ODE for which the given functions $y_1 = e^x$ and $y_2 = xe^x$ are solutions. Verify these two functions are linearly independent.

Answer:

$$y'' - y = 0.$$

Notice that

$$\det \begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix} = e^{2x} \neq 0$$

so they are linearly independent.

Topic: Linear ODE (Existence and uniqueness of solutions, solving homogeneous linear ODE)

Question 3: Find the general form of solution to the following equation.

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$$

Answer:

$$c_1 e^x + c_2 e^{-x} + c_3.$$

Question 4: Consider the following initial value problem.

$$\begin{aligned} \frac{d^2y}{dx^2} - 4y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

(a). Does this initial value problem have a solution? Is the solution unique?

Answer, yes and yes.

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

Answer:

$$\frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh 2x.$$

Topic: Linear ODE system

Question 5: Consider the following initial value problem.

$$\begin{aligned} \frac{dy_1}{dt} &= y_1 + 2y_2 \\ \frac{dy_2}{dt} &= 5y_1 - 2y_2 \\ y_1(0) &= 2 \\ y_2(0) &= 9. \end{aligned}$$

(a). Does this initial value problem have a solution? Is the solution unique?

Answer: Yes

(b). Find the solution to this initial value problem if your answer for part (a) is yes.

Answer:

$$\begin{aligned} y_1 &= 4e^{3t} - 2e^{-4t} \\ y_2 &= 4e^{3t} + 5e^{-4t} \end{aligned}$$

Question 6: Let

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 5 & -4 \end{pmatrix}$$

, eigenvectors: $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftrightarrow 3, \left\{\begin{pmatrix} -\frac{1}{5} \\ -\frac{3}{5} \\ 1 \end{pmatrix}\right\} \leftrightarrow -7$and

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Find the general form of solution to the following system of equations.

$$\frac{dY}{dt} = AY.$$

Answer:

$$Y = c_1 e^t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-7t} \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

Question 7: Consider again the ODE given in Question 3.

(a). Convert this problem into an first-order ODE system.

Answer:

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$$

(b). Solve this ODE system, and compare the solution to the solution you found for Question 3.

Answer:

$$Y = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Understanding questions:

Question 8: What can you say about the set of solutions to a homogeneous linear differential equation?

Question 9: Why do we look at the Wronskian to decide whether a set of functions is linearly independent?

Question 10: If the Wronskian of a set of functions is equal to zero for a particular value of x, does it necessarily mean that the functions are linearly dependent? Why or why not?

Question 11: What condition should the Wronskian of n functions satisfy for these functions to be linearly independent?

Question 12: If you are given a basis of solutions to a linear differential equation, do you know how to write down the general solution?