MATH 322

$\mathcal{TEST}$ 3

April 28th, 2015

Name: ____

Directions:

a. You may NOT use a calculator, your book, or your notes.
b. Do all problems in the spaces provided. If you do run out of space and continue a problem on the back, please indicate this.
c. Show all work. Unless otherwise noted, a solution without work is worth nothing.
d. Circle your answers.
e. Good Luck!

Score:

1.	____________
2.	____________
3.	____________
4.	____________
5.	____________
Total	____________

1. (10pts) Suppose $f(x)$ is a function defined on the interval $-10 < x < 10$ and suppose the Fourier series for $f(x)$ is given by the function $F(x)$ described by

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{10} x + b_n \sin \frac{n\pi}{10} x \right).$$

a. (5pts) Give a formula for the coefficient a_0. Your answer should have $f(x)$ in it and should have an integral in it.

b. (5pts) Give a formula for the coefficient b_3. Your answer should have $f(x)$ in it and should have an integral in it.

2. (15pts)

Recall the sawtooth function, which is the periodic extension of the function

$$f(x) = x + \pi$$

for $-\pi < x < \pi$. The Fourier series (computed in class and in the book) is

$$F(x) = \pi + 2\left(\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right).$$

What is the value of $F(\pi)$? Justify this in two ways: (a) using the convergence theorem for Fourier series and (b) by computing the series directly.

Extra credit (10 points maximum, very little partial credit): Compute the derivative of the Fourier series $F(x)$ by differentiating the series term by term. Does this series converge everywhere? Is this the series of a function that we know?

3.(20pts) Consider the function $g(x) = \sin x$ for $0 \leq x \leq \pi$. In this problem you may find the following formula useful:

$$\int \sin ax \cos bx dx = -\frac{1}{2(a+b)} \cos((a+b)x) - \frac{1}{2(a-b)} \cos((a-b)x).$$

a. (10pts) Compute the Fourier series for the odd half range Fourier expansion for g (the associated series should have period 2π).

b. (10pts) Compute the first two nonzero terms in the Fourier series for the even half range Fourier expansion for g (the associated series should have period 2π).

4. (30pts) Consider the wave equation for the function $u(x,t)$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. In this problem we will find $u(x,t)$ for the string of length $L = 1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, $k = 0.01$) is

$$k\left(\sin \pi x - \frac{1}{2}\sin 2\pi x\right).$$

a. (15pts) Use separation of variables to find the following solutions to the differential equation with given boundary conditions (but not the initial conditions)

$$u_n(x,t) = T_n(t)X_n(x) \;\; = (A_n \cos(n\pi t) + B_n \sin(n\pi t)) \sin(n\pi x).$$

b. (15pts) Find $u(x,t)$ that satisfy the differential equation with boundary conditions and the initial conditions (you may use Part a even if you cannot answer it).

5. (25pts)

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on a metal bar of length π and with $c = 1$ such that one end is kept at temperature $u(0, t) = 0$ and the other has the property that $\frac{\partial u}{\partial x}(\pi, t) = 0$.

a. (10pts) Using separation of variables derive two associated Sturm-Liouville equations, one with boundary conditions and one without boundary conditions.

b. (15pts) Solve the Sturm-Liouville equation with boundary conditions. (5 pts to solve either equation without boundary conditions)