Written Homework 4

1) Find the complex Fourier series of the following functions

$$g(x) = \sin x \qquad -\pi < x < \pi$$
$$h(x) = \begin{cases} 0 & \text{if } -1 < x \leq 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$$

Answer: For $g(x)$, we have that the coefficients are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x \; e^{-inx} dx$$

which you can do directly. Of course, it is easier to remember that

$$\sin x = \frac{1}{2i} \left(e^{ix} - e^{-ix} \right)$$

and so we see that the Fourier series for g is just

$$G(x) = \frac{i}{2} e^{-ix} - \frac{i}{2} e^{ix}.$$

For h, we get

$$\begin{aligned} c_n &= \frac{1}{2} \int_0^1 e^{-inx} dx \\ &= \frac{i}{2n} \left(e^{-in} - 1 \right) \end{aligned}$$

if $n \neq 0$, and $c_0 = \frac{1}{2}$. So we get the Fourier series for h is

$$\sum_{n=-\infty}^{-1} \frac{i}{2n} \left(e^{-in} - 1 \right) e^{inx} + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{i}{2n} \left(e^{-in} - 1 \right) e^{inx}.$$

2) If the complex Fourier series of the function $f(x) = e^x$, where $-\pi < x < \pi$, is given by

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

then give the coefficients c_0, c_1, c_{-1}, c_3.

Answer:

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \frac{e^\pi - e^{-\pi}}{2\pi} = \frac{1}{\pi}\sinh\pi \\
c_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ix} dx = \frac{e^{(1-i)\pi} - e^{-(1-i)\pi}}{2\pi(1-i)} \\
c_{-1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{ix} dx = \frac{e^{(1+i)\pi} - e^{-(1+i)\pi}}{2\pi(1+i)} \\
c_3 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-3ix} dx = \frac{e^{(1-3i)\pi} - e^{-(1-3i)\pi}}{2\pi(1-3i)}
\end{aligned}
$$

3) Convert the differential equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= c^2\frac{\partial^2 u}{\partial x^2} \\
u(x,0) &= f(x)
\end{aligned}
$$

the function $u(x,t)$ to a differential equation for $\hat{u}(w,t)$.

Answer:

$$
\begin{aligned}
\frac{\partial \hat{u}}{\partial t} &= -c^2 w^2 \hat{u} \\
\hat{u}(w,0) &= \hat{f}(w)
\end{aligned}
$$

4) Write an integral form of the solution to the heat equation on the whole real line with initial condition

$$
u(x,0) = \begin{cases} 1 & \text{if } |x| \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

Answer:

$$
u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\int_{-2}^{2} e^{-\frac{(p-x)^2}{4c^2 t}} dp.
$$

5) Find the solution to the heat equation on the whole real line with initial condition $u(x,0) = \delta(x-1) + \delta(x+1)$.

Answer:

$$
u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\left(e^{-\frac{(x-1)^2}{4c^2 t}} + e^{-\frac{(x+1)^2}{4c^2 t}}\right).
$$