Solutions to 6.1 8 a), b), 9 and 6.2 2 b), c), 7, 9

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1 6.1

1.1 number 8

a) Consider the inner product

$$\langle (a,b),(c,d)\rangle = ac - db \text{ on } \mathbb{R}^2$$

Look at

$$\begin{aligned}\langle (a,b),(a,b)\rangle &= a\cdot a - b\cdot b\\ &= a^2 - b^2\end{aligned}$$

$a^2 - b^2 < 0 \implies a^2 < b^2 \implies a < b$
Thus, if $a < b$, then $\langle (a,b),(a,b)\rangle < 0$
So, $\langle (a,b),(c,d)\rangle = ac - bd$ is not an inner product.

b) Consider the inner product

$$\langle A,B\rangle = tr(A+B) \text{ on } \mathbb{M}_{2\times 2}(\mathbb{R})$$

let $\lambda \in \mathbb{F}, \quad C, O \in \mathbb{M}_{2\times 2}(\mathbb{R})$
Then,

$$\begin{aligned}\langle \lambda A + C, B\rangle &= tr(\lambda A + C + B)\\ &= \lambda tr(A) + tr(C) + tr(B)\\ &= \lambda tr(A) + tr(B) + tr(C) + tr(O)\\ &= \langle \lambda A, B\rangle + \langle C, O\rangle \neq \lambda\langle A,B\rangle + \langle C,B\rangle\end{aligned}$$

Thus this is not an inner product because it does not obey linearity.

1.2 number 9

Let β be a basis for finite-dimensional inner product space.

a) Prove that if $\langle x, z\rangle = 0, \forall z \in \beta$, then $x = 0$.

Proof. Suppose $\langle x, z\rangle = 0, \forall z \in \beta$
let $z = \sum_{i=1}^{n} a_i z_i \neq 0$.
Then

$$\begin{aligned}
\langle x, z\rangle &= \langle x, \sum_{i=1}^{n} a_i z_i\rangle \\
&= \sum_{i=1}^{n}\langle x, a_i z_i\rangle \\
&= \sum_{i=1}^{n} \bar{a}_i \langle x, z_i\rangle \\
&= 0, \forall z \in V \iff x = 0
\end{aligned}$$

□

b) Prove that if $\langle x, z\rangle = \langle y, z\rangle, \forall z \in \beta$, then $x = y$.

$$\begin{aligned}
&\implies \langle x, z\rangle - \langle y, z\rangle = 0 \\
&\implies \langle x - y, z\rangle = 0, \forall z \in \beta
\end{aligned}$$

Thus, by part **a)**, $x - y = 0 \implies x = y$

2 6.2

Let $V = \mathbb{P}_2(\mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$
Using $S = \{1, x, x^2\}$ and $h(x) = 1 + x$, build an orthonormal basis for V

Using Gram-Schmidt process we have

$$\begin{aligned}
S' &= \{h_1, h_2, h_3\} \\
h_1 &= h(x) = 1 + x \\
h_2 &= x - \frac{\langle x, h_1\rangle}{\|h_1\|^2} h_1
\end{aligned}$$

$$\|h_1\|^2 = \langle x, 1+x\rangle = \int_0^1 (x+1)^2 dx$$
$$= \int_0^1 x^2 + 2x + 1 dx$$
$$= (\frac{x^3}{3} + x^2 + x)\Big|_0^1$$
$$= \frac{1}{3} + 1 + 1 = \frac{7}{3}$$

$$\langle x, 1+x\rangle = \int_0^1 x + x^2 dx$$
$$= (\frac{x^2}{2} + \frac{x^3}{3})\Big|_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Thus

$$h_2 = x - \frac{(\frac{5}{6})}{(\frac{7}{3})} h_1$$
$$= x - \frac{5}{14}(1+x)$$
$$= x - \frac{5}{14} - \frac{5}{14}x$$
$$= -\frac{5}{14} + \frac{9}{14}x$$

Next,

$$h_3 = x^2 - \frac{\langle x^2, h_1\rangle}{\|h_1\|^2} h_1 - \frac{\langle x^2, h_2}{\|h_2\|^2} h_2$$

Note: $\|h_2\|^2 = \frac{1}{28}$

$$\langle x^2, h_1\rangle = \int_0^1 x^2 + x^3 dx$$
$$= \frac{7}{12}$$

$$\langle x^2, h_2\rangle = \int_0^1 \frac{-5}{14}x^2 + \frac{9}{14}x^3 dx$$
$$= \frac{1}{14}(\frac{-5}{3}x^3 + \frac{9}{4}x^4)\Big|_0^1$$
$$= \frac{1}{24}$$

Now combining all components:

$$\begin{aligned} h_3 &= x^2 - \frac{(\frac{7}{12})}{(\frac{7}{3})}h_1 - \frac{(\frac{7}{12})}{(\frac{1}{28})}h_2 \\ &= x^2 - \frac{1}{4}(1+x) - \frac{49}{3}(-\frac{5}{14} + \frac{9}{14}x) \\ &= x^2 - \frac{1}{4} - \frac{1}{4}x + \frac{35}{6} - \frac{21}{2}x \\ &= x^2 - \frac{43}{2}x + \frac{67}{12} \end{aligned}$$

Thus we have $S' = \{h_1, h_2, h_3\}$, an orthonormal basis for V.

2.1 number 7

let β be a basis for the subspace W of an inner product space V and let $z \in V$. Prove that $z \in W^\perp \iff \langle z, v \rangle = 0$ for all $v \in \beta$.

Proof. ($\Longleftarrow$) Suppose that $\langle z, v \rangle = 0, \forall v \in \beta$
Then $z \in W^\perp$ by definition of orthogonal complement of sets.

($\Longrightarrow$) Now suppose that $z \in W^\perp$.
$\implies z \in V s.t. \langle z, y \rangle = 0, \forall y \in W$. □

2.2 number 9

let $W = span(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. Find an orthonormal basis for W and $W^\perp$.

let $W = \{\frac{(i,0,1)}{\|(i,0,1)\|}\}$.
By definition of $W^\perp$, we have $W^\perp = \{z \in V : \langle z, y \rangle = 0, \forall y \in W\}$
Consider $\langle (i, 0, 1), y \rangle = 0$
if $y = (y_1, y_2, y_3)$, then

$$i \cdot y_1 + 0 \cdot y_2 + 1 \cdot y_3 = 0$$

$$\implies y_1 i + y_3 = 0 \implies y_1 = 1, y_3 = -i$$

or,

$$\implies y_1 = i, y_3 = 1, y_2 = t \in \mathbb{C}$$

Thus, we can make $W^\perp = \{\frac{(i,i,1)}{\|(i,i,1)\|}, \frac{(1,0,-i)}{\|(1,0,-i)\|}\}$