**MATH 413/513**

# $\mathcal{EXAM}$ 1

*October* $14^{th}$, *2015*

**Your Name:** ______________________________

**Directions:**

a. You may NOT use your book or your notes.
b. Please ask for extra scrap paper if needed.
c. Show <u>all</u> work. Unless otherwise noted, a solution without work is worth nothing.
d. Good Luck!

## Score:

| | |
|---|---|
| 1. | ________ |
| 2. | ________ |
| 3. | ________ |
| 4. | ________ |
| 5. | ________ |
| Total | ________ /100 |

**1. (15pts)** Let $V$ be a vector space (not necessarily finite-dimensional) and let $T : V \to V$ be a linear transformation. For each of the following, determine whether or not the condition implies that $T$ is (i) one-to-one, (ii) onto, and/or (iii) an isomorphism (note: an isomorphism from $V$ to itself is called an automorphism). Give short justifications.

**a.** $R(T) = V$.

**b.** $\text{nullity}(T) = 0$ and $V$ is finite dimensional.

**c.** $V$ is finite dimensional and $\text{nullity}(T) + \text{rank}(T) = \dim V$.

**2. (20pts)** Let $V$ be a vector space, let $T : V \to V$ be a linear transformation, and let $T_0 : V \to V$ denote the zero transformation (that is, $T_0(v) = \vec{0}$ for all $v \in V$). Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

**3. (20pts)** Consider the linear transformation $L_A : \mathbb{R}^4 \to \mathbb{R}^5$ determined by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $E_n$ denote the standard basis for $\mathbb{R}^n$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ be an ordered basis of $\mathbb{R}^4$.

**a. (10pts)** Compute $[L_A]_\beta^{E_5}$ .

**b. (10pts)** Compute the change of basis matrix $Q$ taking $\beta$ to $E_4$ and double check that $[L_A]_\beta^{E_6} = AQ$.

**4. (25pts)**

**a. (15pts)** Let $V_1$ and $V_2$ be subspaces of $V$, a finite dimensional vector space. Show that $\dim(V_1 + V_2) \leq \dim V_1 + \dim V_2$

**b. (10pts)** Let $T, U : V \to W$ be linear transformations. Prove that $R(T + U) \subseteq R(T) + R(U)$

**5. (20pts)** An $n \times n$ matrix is *unit upper triangular* if it has ones along the diagonal and zeroes below the diagonal, so it has the form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

**a. (5pts)** Explain why the set of all unit upper triangular matrices does not form a subspace of the vector space of $n \times n$ matrices.

**b. (15pts)** Prove that any unit upper triangular matrix is invertible.