## $\mathcal{EXAM} \ 1$

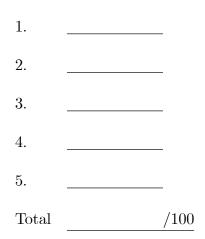
## $October\; 14^{th},\; 2015$

Your Name: \_\_\_\_\_

## **Directions:**

- a. You may NOT use your book or your notes.
- b. Please ask for extra scrap paper if needed.
- c. Show <u>all</u> work. Unless otherwise noted, a solution without work is worth nothing.
- d. Good Luck!

Score:



**1.** (15pts) Let V be a vector space (not necessarily finite-dimensional) and let  $T: V \to V$  be a linear transformation. For each of the following, determine whether or not the condition implies that T is (i) one-to-one, (ii) onto, and/or (iii) an isomorphism (note: an isomorphism from V to itself is called an automorphism). Give short justifications.

**a.** R(T) = V.

**b.** nullity (T) = 0 and V is finite dimensional.

**c.** V is finite dimensional and nullity  $(T) + \operatorname{rank}(T) = \dim V$ .

**2.** (20pts) Let V be a vector space, let  $T : V \to V$  be a linear transformation, and let  $T_0: V \to V$  denote the zero transformation (that is,  $T_0(v) = \vec{0}$  for all  $v \in V$ ). Prove that  $T^2 = T_0$  if and only if  $R(T) \subseteq N(T)$ .

3. (20pts) Consider the linear transformation  $L_A : \mathbb{R}^4 \to \mathbb{R}^5$  determined by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
  
Let  $E_n$  denote the standard basis for  $\mathbb{R}^n$  and  $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$  be an ordered basis of  $\mathbb{R}^4$ .

**a.** (10pts) Compute  $[L_A]_{\beta}^{E_5}$ .

**b.** (10pts) Compute the change of basis matrix Q taking  $\beta$  to  $E_4$  and double check that  $[L_A]^{E_6}_{\beta} = AQ.$ 

## 4. (25pts)

a. (15pts) Let  $V_1$  and  $V_2$  be subspaces of V, a finite dimensional vector space. Show that  $\dim (V_1 + V_2) \leq \dim V_1 + \dim V_2$ 

**b.** (10pts) Let  $T, U: V \to W$  be linear transformations. Prove that  $R(T+U) \subseteq R(T) + R(U)$ 

5. (20pts) An  $n \times n$  matrix is *unit upper triangular* if it has ones along the diagonal and zeroes below the diagonal, so it has the form

$$\begin{pmatrix}
1 & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & 1 & a_{23} & \cdots & a_{2n} \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 & a_{(n-1)n} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

**a.** (5pts) Explain why the set of all unit upper triangular matrices does not form a subspace of the vector space of  $n \times n$  matrices.

b. (15pts) Prove that any unit upper triangular matrix is invertible.