$\mathcal{EXAM}\ 2$

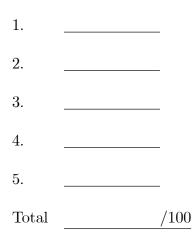
November 18th, 2015

Your Name: _____

Directions:

- a. You may NOT use your book or your notes.
- b. Please ask for extra scrap paper if needed.
- c. Show <u>all</u> work. Unless otherwise noted, a solution without work is worth nothing.
- d. The total possible points are 105, but your score will be counted out of 100.
- e. Good Luck!

Score:



1. (20pts) Suppose A is a 10×10 matrix with entries in \mathbb{C} and the rank of A is 7. For the following, determine whether the statement is necessarily true, necessarily false, or there is not enough information to determine if it is true or false. Give short justifications.

a. det A = 0

b. The reduced row echelon form of A contains at least one row of zeroes.

c. There exists an invertible matrix Q such that $Q^{-1}AQ$ is diagonal.

d. There exist invertible matrices Q and P such that QAP is diagonal.

e. A is a product of elementary matrices.

2. (20pts) Consider the matrix $A = \begin{pmatrix} 1 & -3 & -7 & 5 & -4 \\ 2 & 2 & 2 & 6 & -8 \\ -1 & 2 & 5 & -2 & -1 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Answer the following questions about A:

a. What is the rank of A?

b. Give a basis for the column space of A.

c. Give a basis for the nullspace of A.

d. Suppose for some vector $b \in \mathbb{R}^4$, $x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ is a solution to Ax = b. Find all solutions of Ax = b in \mathbb{R}^5 .

3. (20pts)

a. (10pts) Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

b. (10pts) Suppose a matrix A is diagonalizable, so there is an invertible matrix Q such that $Q^{-1}AQ$ is diagonal. Show that for each standard basis element e_i , the vector Qe_i is an eigenvector for A. (Hint: each e_i is an eigenvector for the diagonal matrix.)

4. (25pts) Let T be an invertible linear operator on a finite-dimensional vector space V. a. (15pts) Prove that the scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1} . Hint: Notice that $T - \lambda I_V = -\lambda T (T^{-1} - \lambda^{-1} I_V)$.

b. (10pts) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1} (Note: you may use the result of part a even if you cannot prove it.)

5. (20pts) Recall that an real inner product space is a vector space V over \mathbb{R} together with a product $\langle \cdot, \cdot \rangle$ such that for all $x, y, z \in V$ and $c \in \mathbb{R}$,

- $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$,
- $\langle cx, z \rangle = c \langle x, z \rangle$,
- $\langle x, z \rangle = \langle z, x \rangle$,
- $\langle x, x \rangle > 0$ if $x \neq \vec{0}$.

Consider the vector space $\mathbb{R}^{2\times 2}$ of two-by-two matrices with real entries and define a product $\langle A, B \rangle = tr(B^T A)$, where tr denotes the trace (recall that $tr\begin{pmatrix}a & b \\ c & d\end{pmatrix} = a + d$). Show that this is an inner product space. You may use the following facts without proof:

- 1. $\mathbb{R}^{2\times 2}$ is a vector space over \mathbb{R} .
- 2. The trace is linear: tr(cA+B) = c tr(A) + tr(B) for all $c \in \mathbb{R}$ and $A, B \in \mathbb{R}^{2 \times 2}$.
- 3. The trace is invariant under transpose: $tr(A^T) = tr(A)$ for all $A \in \mathbb{R}^{2 \times 2}$.
- 4. $(AB)^T = B^T A^T$ for all $A, B \in \mathbb{R}^{2 \times 2}$.