Math 443/543 Graph Theory Notes

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September 3, 2008

1 Introduction

We will begin by considering several problems which may be solved using graphs, directed graphs (digraphs), and networks.

Problem 1: The Königsberg Bridge Problem. The city of Königsberg has a river with two islands and seven bridges. The question is whether one can traverse each bridge exactly once. This is a classical problem solved by Euler in the 1700s

Problem 2: Coloring maps. Given a map of a continent, how many colors does one need to color each country so that it has a different color than each of its neighbors? It turns out the answer is 4, but the proof is extremely difficult. We will prove that 5 is sufficient, which is a weaker result.

Problem 3: Ranking pages on the internet. The internet is a (directed) graph. How can one rank the importance of all pages on the internet? One answer is the PageRank algorithm, used by google for returning results of internet searches.

These are just some of the problems we will investigate in the course of the semester. The formalism of graphs, directed graphs, and networks will serve as a way to model many different situations, and then we will develop techniques devoted to these abstract models. This is very much like when you learn to solve problem using algebra or calculus by converting the problems into equations, and then learning to solve the equations.

2 This course

See the syllabus for information about this course.

3 What is a graph, digraph, and a network

Definition 1 (Chartrand) *A graph G is a finite, nonempty set V together with a relation R on V which is:*

1. *irreflexive, i.e., $(v, v) \notin R$ for any $v \in V$, and*

2. symmetric, i.e., $(v,w) \in R$ if and only if $(w,v) \in R$.

Note: a relation R on a set V is a subset of $V \times V$ (ordered pairs of elements in V).

Definition 2 (Bondy and Murty) *A graph G is a triple $(V(G), E(G), \psi_G)$ where $V(G)$ and $E(G)$ are disjoint sets and*

$$\psi_G : E(G) \to Sym(V(G), V(G))$$

is a map from $E(G)$ to unordered pairs of vertices. The map ψ_G is called the incidence map.

Often we will consider $G = (V, E)$, where in this case, we identify E and $\psi_G(E)$.

Definition 3 *In these two definitions, elements of V are called* vertices *and elements of E are called* edges*.*

Why are these definitions the same? Given the second definition, we simply let $V = V(G)$ and

$$R = \bigcup_{e \in E(G)} [\psi_G(e)]_1 \cup [\psi_G(e)]_2,$$

where $[\psi_G(e)]_1$ and $[\psi_G(e)]_2$ are the two ways to order the unordered pairs. Given the first definition, we simply let $E(G)$ be the unordered pairs corresponding to elements of R, and let ψ_G be, essentially, the identity.

BUT: there is a difference. Bondy and Murty allow for multiple edges between the same vertices and for edges which start and end at the same vertex! This is an example of how the definition of a graph is not entirely consistent from one place to the next and one must be careful.

Some graph terminology:

Definition 4 *The order of a graph is the number of vertices, i.e., $|V| = |V(G)|$.*

Definition 5 *The size of a graph is the number of edges, i.e., $|E| = |E(G)|$*

We will usually denote edges as uv, with the understanding that $uv = vu$. In the Chartrand definition, this means that both (u,v) and (v,u) are in R.

Note, it is possible for the size to be zero, but not the order!

We often denote a graph by a diagram, and will often refer to the diagram as the graph itself. Notice that two different diagrams may correspond to the same graph!!

In general, I prefer the description in BM, but will often assume the graph is entirely given by the vertices and edges, unless we are interested graphs with multiple edges between the vertices, in which case we will need the whole definition. We will also generally assume that graphs do not have loops (edges which begin and end at the same vertex).

Some more terminology:

Definition 6 *Suppose $u, v, w \in V$ and $uv = e \in E$ and $uw \notin E$. Then:*

- *We say e* joins *u and v.*
- *We say u and v are* adjacent *vertices and u and w are* nonadjacent *vertices.*
- *We say that u and v are* incident *with (or to or on) e.*
- *If $uv' = e' \in E$ and $v \neq v'$, then we say that e and e' are* adjacent *edges.*

For completeness now, we define digraphs and networks.

Definition 7 *A* directed graph, or digraph*, is a vertex set together with an irreflexive relation. This is the same as saying that edges are ordered pairs instead of unordered pairs.*

Note, if a digraph is symmetric, it can be represented as a graph.

Definition 8 *A* network *is a graph G together with a map $\phi : E \to \mathbb{R}$. The function ϕ may represent the length of an edge, or conductivity, or cross-sectional area or many other things.*

4 Isomorphism of graphs

4.1 Equivalence relations

Studying all possible graphs is a daunting task, since it would require us to consider all finite sets! A usual technique in mathematics is to partition large sets like this using an equivalence relation. See A.3 in Chartrand for this material.

Definition 9 *An equivalence relation R on a set S is a subset of $S \times S$ satisfying:*

1. *reflexivity: $(x, x) \in R$ for every $x \in S$,*
2. *symmetry: $(x, y) \in R$ if and only if $(y, x) \in R$, and*
3. *transitivity: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.*

Note that sometimes denote $(x, y) \in R$ by $x\ R\ y$ or $x \sim y.'$

Example 10 *The relation $<$ on the set $\mathbb{R}$ is not an equivalence relation because it is neither reflexive or symmetric (it is transitive!)*

Example 11 *The relation $\leq$ on the set $\mathbb{R}$ is not an equivalence relation because it is not symmetric (it is reflexive and transitive).*

Example 12 *The relation $=$ on the set $\mathbb{R}$ is an equivalence relation.*

Example 13 *The relation "is equal modulo 2" on the set $\mathbb{Z}$ of integers is an equivalence relation.*

Example 14 *The relation defining a graph is not an equivalence relation since it is not reflexive and may not be transitive.*

Example 15 *The relation "is parallel to" on the set of lines in the plane is an equivalence relation.*

The importance of equivalence relations is that they partition the set S into pieces.

Definition 16 *The equivalence class of x, usually written as $[x]$ or $[x]_R$, is defined as*

$$[x] = \{y \in S : (x, y) \in R\}.$$

Note that $[x] = [y]$ if and only if $(x, y) \in R$, i.e., x is equivalent to y. The importance of equivalence relations is that they form a partition.

Definition 17 *Let S be a set and let $A_1, A_2, \ldots, A_n$ be nonempty, disjoint subsets of S. The set $P = \{A_1, \ldots, A_n\}$ is a* partition *if $S = A_1 \cup \cdots \cup A_n$.*

Example 18 *The set $\{E, O\}$ where E are even integers and O are odd integers is a partition of the integers $\mathbb{Z}$.*

Example 19 *Consider $S = \{1, 2, 3\}$. The following are partitions of S: $P_1 = \{\{1\}, \{2\}, \{3\}\}$, $P_2 = \{\{1, 3\}, \{2\}\}$, $P_3 = \{\{1, 2, 3\}\}$. The following sets are not partitions: $Q_1 = \{\{1\}, \{2\}\}$, $Q_2 = \{\{1, 2\}, \{2, 3\}\}$, $Q_3 = \{\{1\}, \{1, 2, 3\}\}$.*

Proposition 20 *For any equivalence relation R on a set S, the set of equivalence classes*

$$P = \{[x] : x \in S\}$$

form a partition.

Note that we mean P contains one copy of the equivalence class $[x]$, not one for each $x \in S$.

Proof. We need to show that the equivalence classes are nonempty, disjoint and that their union is all of S. The latter is clearly true, since

$$S = \bigcup_{x \in S} [x].$$

By reflexivity, we know that $x \in [x]$, so equivalent classes are nonempty. In order to show they are disjoint, we see that if $y \in [x] \cap [x']$, then $(x, y) \in R$ and $(y, x') \in R$, so by transitivity, $(x, x') \in R$ and hence $[x] = [x']$. Thus, for any two elements, their equivalence classes are either the same or disjoint. ■

4.2 Isomorphism

We are now ready to consider a particular equivalence relation on graphs, isomorphism. We would like it to encapsulate when two graphs which are technically different should have all of the same "graph theoretic" properties.

Definition 21 *The graphs $G = (V, E)$ and $G' = (V', E')$ are* isomorphic *if there is a bijection*

$$\phi : V \to V'$$

such that

$$(v, w) \in E \text{ if and only if } (\phi(v), \phi(w)) \in E'.$$

The map ϕ is called an isomorphism*.*

Remark 22 *A map $\phi : A \to B$ is a bijection if there exists another map $\phi^{-1} : B \to A$ such that $\phi^{-1} \circ \phi : A \to A$ is the identity and $\phi \circ \phi^{-1} : B \to B$ is the identity. It is well-known that ϕ is a bijection if and only if it is*

1. *one-to-one (or injective): if $\phi(x) = \phi(y)$ then $x = y$, and*
2. *onto (or surjective): $\phi(A) = B$.*

For more information, see Chartrand A.4.

Remark 23 *In the BM definition, one may have multiple edges with the same vertices, so the above definition is not sufficient. Instead, we need to have two bijections, $\phi_V : V \to V'$ and $\phi_E : E \to E'$, such that*

$$\psi_{G'}(\phi_E(e)) = \phi_V(u)\ \phi_V(v) \text{ if and only if } \psi_G(e) = uv.$$

Show examples.

Proposition 24 *The relation "is isomorphic to" is an equivalence relation.*

Proof. We need to show the following:

1. Reflexive: yes, G is isomorphic to G taking the identity map as ϕ.
2. Symmetric: yes, if ϕ is an isomorphism from G to G', it is a bijection and hence has an inverse, and thus ϕ^{-1} is an isomorphism from G' to G.
3. Transitive: yes, if ϕ is an isomorphism from G to G' and ϕ' is an isomorphism from G' to G''. Then we claim that $\phi'' = \phi' \circ \phi$ is an isomorphism from G to G''. Certainly, it is a bijection, since it has inverse $(\phi'')^{-1} = \phi^{-1} \circ (\phi')^{-1}$. Also, we see that $(v, w) \in E$ if and only if $(\phi(v), \phi(w)) \in E'$ if and only if $(\phi' \circ \phi(v), \phi' \circ \phi(w)) = (\phi''(v), \phi''(w)) \in E''$.

■

In order to show two graphs are isomorphic, one must find an isomorphism. How do we show two graphs are not isomorphic?

4.3 Easy invariants of graphs

We are looking for properties which are preserved by isomorphism. That way, if we find two graphs with different such properties, they cannot be isomorphic, otherwise the isomorphism would imply that they have the same property!

The first such property is the degree.

Definition 25 *The* degree *of a vertex v, denoted* $\deg_G v$ *or* $\deg v$, *is the number of edges incident with v.*

Examples of graphs with vertices of different degrees.

Proposition 26 *If $\phi : G \to G'$ is an isomorphism, then $\deg_G v = \deg_{G'} \phi(v)$ for each $v \in V(G)$.*

Note this implies that the set of vertex degrees for each graph must be the same.

Proof. Since ϕ is an isomorphism, we see that for each edge $(v, w) \in E(G)$, where v is the given vertex and w varies, there is an edge $(\phi(v), \phi(w)) \in E(G')$, and thus

$$\deg_G v \leq \deg_{G'} \phi(v).$$

Using the isomorphism ϕ^{-1}, we see that

$$\deg_{G'} v' \leq \deg_G \phi^{-1}(v')$$

for any $v' \in V(G')$. Taking $v' = \phi(v)$ (which we can do since ϕ is a bijection!) we see that

$$\deg_{G'} \phi(v) \leq \deg_G v.$$

Thus, we must have that

$$\deg_{G'} \phi(v) = \deg_G v.$$

■

Show examples of non-isomorphic graphs using degree.

Show example of nonisometric graphs with the same degree (see C, Fig. 2.5).

Definition 27 *A (p, q)-*graph *is a graph with order p and size q.*

Theorem 28 *If G is a (p, q)-graph and $V(G) = \{v_1, \ldots, v_p\}$, then*

$$\sum_{i=1}^{p} \deg_G(v_i) = 2q.$$

Proof. In the left sum, for each vertex we count the edges incident on it. Since each edge is incident on two vertices and each vertex is counted, we get the result. ■

Note that this is quite a restriction on the degrees of vertices, i.e., we cannot just assign degree numbers arbitrarily and hope that there is a graph with that collection of degrees.

Definition 29 *We say a vertex is* even *or* odd *depending on whether its degree is even or odd.*

Theorem 30 *Every graph has an even number of odd vertices.*

Proof. If there are no odd vertices, then there are an even number of odd vertices. Suppose $V(G) = \{v_1, \ldots, v_p\}$, and suppose the vertices $\{v_1, \ldots, v_n\}$ are odd and $\{v_{n+1}, \ldots, v_p\}$ are even (there may be none). We look at the theorem, and see that

$$\sum_{i=1}^{n} \deg_G v_i + \sum_{j=n+1}^{p} \deg_G v_j = 2q.$$

Recall that the sum of an even number of odd numbers is even and the sum of an odd number of odd numbers is odd. any sum of even numbers is even. Thus we get that

$$\sum_{i=1}^{n} \deg_G v_i = 2q - \sum_{j=n+1}^{p} \deg_G v_j$$

has an even number on the right and an odd number on the left if n is odd. Thus n must be even. ■

Remark 31 *How does one prove that the sum of an odd number of odd numbers is odd? One must use induction and we will do it in the homework.*

Definition 32 *If every vertex has degree* r, *we say that the graph is* r-regular. *A graph is* complete *if every edge is connected to every other edge, or, equivalently, it is* $(p-1)$*-regular where* p *is the order.*

Draw some complete graphs.

4.4 Enumeration of graphs by order and size

We may try to enumerate the isomorphism classes of graphs by their order and size. Since degree is an invariant, we see that there is only one graph of order 1 (a single vertex), which has size zero. This means that given two graphs G_1 and G_2 of size 1, they must be isomorphic. There is one graph with (order,size)=$(2,0)$ and one graph with $(2,1)$. There are three $(4,3)$ graphs.

One can use the pigeonhole principle to see that given any 4 graphs of type $(4,3)$, two must be isomorphic. The pigeonhole principle can be stated as follows. In the sequel, $\lceil x \rceil$ is the smallest integer greater than x, also called the ceiling of x.

Proposition 33 (Pigeonhole Principle) *If* $\{S_1, S_2, \cdots, S_k\}$ *is a partition of the set* S *containing* n *elements, then there must be at least one* i *such that* S_i *contains* $\lceil n/k \rceil$ *elements.*

Proof. Suppose each S_i contain fewer than $\lceil n/k \rceil$ elements. First suppose n/k is an integer. Then the total number of elements in

$$\bigcup_{i=1}^{k} S_i$$

is less than $k \lceil n/k \rceil = n$. Hence the union is a strict subset of S, and thus the set is not a partition. If n/k is not an integer, then each set contains at most $\lceil n/k \rceil - 1 = (n-m)/k$ elements, where $1 \leq m < k$. Thus the total number of elements is less than or equal to $(n-m)$, and thus the same argument applies. ■

Using the Pigeonhole Principle, we see that since there are 4 graphs of type $(4,3)$ up to isomorphism, which means that we have a partition of the set with $k=4$, then if we have a set of 5 graphs, at least one element of the partion (isomorphism class) must contain $\lceil 5/4 \rceil = 2$ elements.

5 Connected graphs

An important type of graph is a connected graph. In this section we will show the following:

Theorem 34 *Let G be a connected graph. Any edge $e \in E(G)$ is a bridge of G if and only if it is not contained in any cycle.*

Definition 35 *A* subgraph *H of a graph G is a graph such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.*

Remark 36 *Note that we require that H be a graph, so this places restrictions on $V(H)$ and $E(H)$, i.e., you cannot just take arbitrary subsets of $V(G)$ and $E(G)$.*

Definition 37 *The following are ways to traverse a graph:*

1. *A* uv-walk *is an alternating sequence of vertices and edges starting at u and ending at v such that adjacent elements of the sequence are incident. E.g., $u, uw_1, w_1, w_1w_2, w_2, w_2v, v$.*

2. *A uv-*trail *is a uv-walk which does not repeat any edge. It may repeat a vertex, though.*

3. *A uv-*path *is a uv-trail which does not repeat any vertex.*

Definition 38 *Two vertices u and v are* connected *if $u = v$ or there exists a uv-path. A* graph is connected *if every pair of vertices are connected. Otherwise, we say that the graph is* disconnected.

Remark 39 *If there exists a uv-walk or a uv-trail, then we also see that u and v are connected. This is because, given a uv-walk, we can construct a uv-path in the following way. Given a uv-walk, if any vertex is visited more than once, we may remove the part of the walk between the first time the vertex is visited and the last time. In this way, we may convert any uv-walk into a uv-path (if an edge is traversed twice, then a vertex must be visited more than once).*

Definition 40 *A subgraph H of G is a component if is not contained in any connected subgraph.*

Definition 41 *A uv-trail in which $u = v$ which contains at least three edges is called a* circuit*. A circuit which does not repeat any vertices is called a* cycle*.*

Remark 42 *Sometimes we will be lax in the definition of walk, trail, path, etc. by labeling only the vertices or only the edges in the walk, etc.*

Definition 43 *An edge e in a connected graph G is a bridge if $G - e$ is disconnected.*

Note: $G - e$ is the graph gotten by removing the edge e from the graph.

Remark 44 *There is also the notion of a cut vertex, a vertex v such that $G - v$ is disconnected. Here, $G - v$ is the graph gotten by removing both the vertex v and all edges incident on v. Note that $G - v$ may have many components, whereas $G - e$ has either two components if e is a bridge or one component if it is not.*

We are now ready to prove the theorem

Proof of Theorem 34. First suppose $e = uv \in E(G)$ is contained in a cycle. Then clearly u and v are connected in $G - e$ since we take the complement of e in the cycle. If we show that "x is connected to y in G" is an equivalence relation, then this implies that e is not a bridge: Since x is connected to y in G, there is a path P from x to y. If P does not contain e, then x is connected to y in $G - e$. If P does contain e, we know that x is connected to u and y is connected to v or vice versa. But since u is connected to v in $G - e$, then we have that x is connected to y.

Now suppose that e is not contained in any cycle. That implies that there is no path from u to v in $G - e$ since otherwise, the concatenation with e would be a cycle. Thus e is a bridge. ■