Math 443/543 Graph Theory Notes 12: Laplacians on Graphs

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1 Approximating the Laplacian on a lattice

Recall that the Laplacian is the operator

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

acting on functions $f(x, y, z)$, with analogues in other dimension. Let's first consider a way to approximate the one-dimensional Laplacian. Suppose $f(x)$ is a function and I want to approximate the second derivative $\frac{d^2 f}{dx^2}(x)$. We can take a centered difference approximation to the get this as

$$\begin{aligned}
\frac{d^2 f}{dx^2}(x) &\approx \frac{f'\left(x + \frac{1}{2}\Delta x\right) - f'\left(x - \frac{1}{2}\Delta x\right)}{\Delta x} \\
&\approx \frac{1}{\Delta x}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x) - f(x) + (f(x-\Delta x) - f(x))\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x) + f(x-\Delta x) - 2f(x)\right]
\end{aligned}$$

Note that if we take $\Delta x = 1$, then this only depends on the value of the function at the integer points.

Now consider the graph consisting of vertices on the integers of the real line and edges between consecutive integers. Give an function f on the vertices, we can compute the Laplacian as

$$\triangle f(v_i) = f(v_{i+1}) + f(v_{i-1}) - 2f(v_i)$$

for any vertex v_i. Notice that the Laplacian is an infinite matrix of the form

$$\triangle f = \begin{pmatrix} \cdots & \cdots & & & & & \\ \cdots & -2 & 1 & 0 & & & \\ & 1 & -2 & 1 & 0 & & \\ & 0 & 1 & -2 & 1 & 0 & \\ & & 0 & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & \cdots \\ & & & & & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ f(v_2) \\ f(v_1) \\ f(v_0) \\ f(v_{-1}) \\ f(v_{-2}) \\ \cdots \end{pmatrix}.$$

Also note that that matrix is exactly equal to the adjacency matrix minus twice the identity. The number 2 is the degree of each vertex, so we can write the matrix, which is called the Laplacian matrix, as

$$L = A - D$$

where A is the adjacency matrix and D is the diagonal matrix consisting of degrees (called the degree matrix).

Note that something similar can be done for a two-dimensional grid. We see that

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) &\approx \frac{f_x\left(x + \frac{1}{2}\Delta x, y\right) - f_x\left(x - \frac{1}{2}\Delta x, y\right)}{\Delta x} + \frac{f_y\left(x, y + \frac{1}{2}\Delta y\right) - f_y\left(x, y - \frac{1}{2}\Delta y\right)}{\Delta y} \\
&\approx \frac{1}{\Delta x}\frac{f(x+\Delta x, y) - f(x,y) - [f(x,y) - f(x - \Delta x, y)]}{\Delta x} \\
&+ \frac{1}{\Delta y}\frac{f(x, y+\Delta y) - f(x,y) - [f(x,y) - f(x, y - \Delta y)]}{\Delta y} \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x, y) - f(x,y) + (f(x-\Delta x, y) - f(x,y))\right] \\
&+ \frac{1}{(\Delta y)^2}\left[f(x, y+\Delta y) - f(x,y) + (f(x, y-\Delta y) - f(x,y))\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x, y) + (f(x-\Delta x, y) - 2f(x,y))\right] \\
&+ \frac{1}{(\Delta y)^2}\left[f(x, y+\Delta y) + (f(x, y-\Delta y) - 2f(x,y))\right].
\end{aligned}$$

If we let $\Delta x = \Delta y = 1$, then we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \approx f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y).$$

Note that on the integer grid, this translates to the sum of the value of f for the four vertices neighboring the vertex, minus 4 times the value at the vertex. This is precisely the same as the last time, and we see that this operator can again be written as

$$L = A - D.$$

In general we will call this matrix the Laplacian matrix. It can be thought of as a linear operator on functions on the vertices. Sometimes the Laplacian will denote the negative of this operator (which gives positive eigenvalues instead of negative ones), and sometimes a slight variation is used in the graph theory literature.

2 Electrical networks

One finds applications of the Laplacian in the theory of electrical networks. Recall that the current through a circuit is proportional to the change in voltage, and that constant of proportionality is called the conductance (or resistance, depending on where the constant is placed. Thus, for a wire with conductance C between points v and w with voltages $f(v)$ and $f(w)$ respectively, the current from v to w is $C(f(v) - f(w))$. Kirchoff's law says that if we have a network of wires, each with conductance C, the total current through any given point is zero. Thus, we get that

$$C \sum_{v \text{ adjacent to } w} (f(w) - f(v)) = 0$$

which is the same as $\triangle f = 0$. Note that if the conductances are different, then we get an equation like

$$\sum_{v \text{ adjacent to } w} c_{vw} (f(w) - f(v)) = 0,$$

which is quite similar to the Laplacian. Hence we can use the Laplacian to understand graphs by attaching voltages to some of the vertices and seeing what happens at the other vertices. This is very much like solving a boundary value problem for a partial differential equation!

3 Spectrum

This matrix is symmetric, and thus it has a complete set of eigenvalues. The set of these eigenvalues is called the spectrum of the Laplacian. Notice the following.

Proposition 1 *Let G be a finite graph. The eigenvalues of the matrix L are all nonpositive. Moreover, the constant vector $\vec{1} = (1, 1, 1, \ldots, 1)$ is an eigenvector with eigenvalue zero.*

Proof. It is clear that $\vec{1}$ is an eigenvector with eigenvalue 0 since the sum of the entries in each row must be zero. Now, notice that we can write

$$\begin{aligned}
v^T Lv &= \sum v_i \left(Lv\right)_i \\
&= \sum_i v_i \sum_j L_{ij} v_j \\
&= \sum_{i,j} v_i \left(v_j - v_i\right) \\
&= \frac{1}{2}\left[\sum_{i,j} v_i \left(v_j - v_i\right) + \sum_{i,j} v_j \left(v_i - v_j\right)\right] \\
&= -\frac{1}{2}\sum_{i,j} \left(v_i - v_j\right)^2 \leq 0.
\end{aligned}$$

Now note that if v is an eigenvector of L with eigenvalue λ, then $Lv = \lambda v$, and

$$v^T Lv = \lambda v^T v = \lambda \sum_i v_i^2.$$

Thus we have that

$$\lambda = \frac{-\frac{1}{2}\sum_{i,j}\left(v_i - v_j\right)^2}{\sum_i v_i^2} \leq 0.$$

■

Definition 2 *The eigenvalues of $-L$ can be arranged $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{p-1}$, where p is the order of the graph. The collection $(\lambda_0, \lambda_1, \ldots, \lambda_p)$ is called the* spectrum *of the Laplacian.*

Remark 3 *Sometimes the Laplacian is taken to be $D^{-1/2}LD^{-1/2}$. If there are no isolated vertices, these are essentially equivalent.*

Remark 4 *Note that the Laplacian matrix, much like the adjacency matrix, depends on the ordering of the vertices and must be considered up to conjugation by permutation matrices. Since eigenvalues are independent of conjugation by permutation matrices, the spectrum is an isomorphism invariant of a graph.*

Here are some easy facts about the spectrum.

Definition 5 *Let G be a directed (p,q)-graph. The oriented vertex-edge incidence graph is a $p \times q$ matrix $Q = [q_{ir}]$, such that $q_{ir} = 1$ if $e_r = (v_i, v_j)$ for some j and $q_{ir} = -1$ if $e_r = (v_j, v_i)$ for some j.*

Proposition 6 *The Laplacian matrix L satisfies*

$$-L = QQ^T$$

for any oriented vertex-edge incidence graph (so, given an undirected graph, we can take any orientation), i.e.,

$$L_{ij} = -\sum_{r=1}^{q} q_{ir} q_{jr}.$$

Proof. If $i = j$, then we see that $L_{ii} = -\deg v_i$. Notice that q_{ir} is nonzero if v_i is in edge e_r. Thus

$$\sum_{r=1}^{q} (q_{ir})^2 = \deg v_i.$$

Now for $i \neq j$, we have that $q_{ir} q_{jr} = -1$ if $e_r = v_i v_j$, giving the result. ■

Remark 7 *This observation can be used to give a different proof that L_{ij} has all nonpositive eigenvalues.*

Proposition 8 *For a graph G of order p,*

$$\sum_{i=0}^{p-1} \lambda_i = 2q.$$

Proof. The sum of the eigenvalues is equal to the trace, which is the sum of the degrees. ■

We will be able to use the eigenvalues to determine some geometric properties of a graph.

4 Connectivity and spanning trees

Recall that $\lambda_0 = 0$, which means that the matrix L is singular and its determinant is zero. Recall the definition of the adjugate of a matrix.

Definition 9 *If M is a matrix, the adjugate is the matrix $M^\dagger = \left[M_{ij}^\dagger\right]$ where $M_{ij}^\dagger$ is equal to $(-1)^{i+j} \det\left(\hat{M}_{ij}\right)$, where $\hat{M}_{ij}$ is the matrix with the ith row and jth column removed.*

The adjugate has the property that

$$MM^\dagger = (\det M)\, I,$$

where I is the identity matrix. Applying this to L gives that

$$LL^\dagger = 0.$$

Now, the $p \times p$ matrix L has rank less than p. If it is less than or equal to $p-2$, then all determinants of $(p-1) \times (p-1)$ submatrices are zero, and hence $L^\dagger = 0$. If L has rank $p-1$, then it has only one zero eigenvalue, which must be $(1, 1, \ldots, 1)^T$. Since $LL^\dagger = 0$, all columns of $L^\dagger$ must be a multiple of $(1, 1, \ldots, 1)^T$. But L is symmetric, so that means that $L^\dagger$ must be a multiple of the matrix of all ones. This motivates the following definition.

Definition 10 *We define $t(G)$ by*

$$t(G) = (-1)^{i+j} \det \left(-\hat{L}_{ij} \right)$$

for any i and j (it does not matter since all are the same).

Note that $t(G)$ is the product $\frac{1}{p}\lambda_1\lambda_2\cdots\lambda_{p-1}$. Also note that $t(G)$ is an integer.

Recall that a spanning tree of G is a subgraph containing all of the vertices of G and is a tree.

Theorem 11 *The number $t(G)$ is equal to the number of spanning trees of G.*

Proof. Omitted, for now. ■

We can apply this, however, as follows.

Example 1, consider the graph K_3. Clearly this has Laplacian matrix

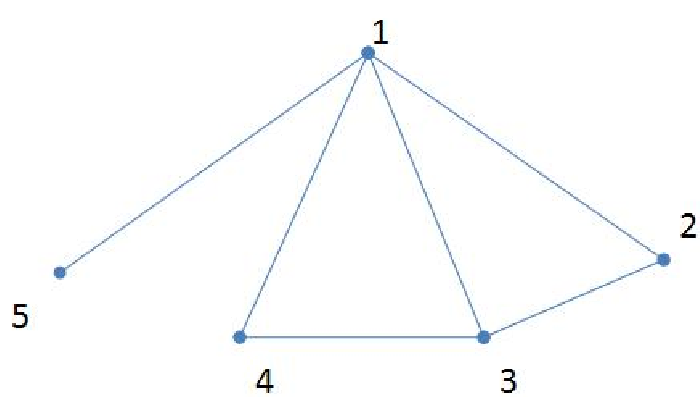
$$L(G) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The number of spanning trees are equal to

$$\det \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 3.$$

It is clear that each spanning tree is given by omitting one edge, so it is clear there are 3.

Example 2: Consider the following graph.



Its Laplacian matrix is

$$L(G) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The number of spanning trees are equal to

$$\begin{aligned} t(G) &= \det \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \\ &= 2(6-1) + (-2) = 8 \end{aligned}$$

One can check directly that it has eight spanning trees.

Corollary 12 $\lambda_1 \neq 0$ *if and only if* G *is connected.*

Proof. $\lambda_1 = 0$ if and only if $t(G) = 0$ since $t(G)$ is the product of the eigenvalues $\lambda_1 \lambda_2 \cdots \lambda_{p-1}$ and λ_1 is the minimal eigenvalue after λ_0. But $t(G) = 0$ means that there are no spanning trees, so G is not connected. ■

Now we can consider the different components.

Definition 13 *The disjoint union of two graphs* $G = G_1 \sqcup G_2$ *is the graph gotten by taking* $V(G) = V(G_1) \sqcup V(G_2)$ *and* $E(G) = E(G_1) \sqcup E(G_2)$ *where* $\sqcup$ *is the disjoint union of sets.*

It is not hard to see that if we number the vertices in G by first numbering the vertices of G_1 and then numbering the vertices of G_2, that the Laplacian matrix takes the form

$$L(G) = \begin{pmatrix} L(G_1) & 0 \\ 0 & L(G_2) \end{pmatrix}.$$

This means that the eigenvalues of $L(G)$ are the union of the eigenvalues of $L(G_1)$ and $L(G_2)$. This implies the following.

Corollary 14 *If* $\lambda_n = 0$, *then there are at least* $n+1$ *connected components of* G.

Proof. Induct on n. We already know this is true for $n = 1$. Suppose $\lambda_n = 0$. We know there must be at least n components, since $\lambda_n = 0$ implies $\lambda_{n-1} = 0$. We can then write the matrix $L(G)$ in the block diagonal form with $L(G_i)$ along the diagonal for some graphs G_i. Since $\lambda_n = 0$, one of these graphs must have $\lambda_1(G_i) = 0$. But that means that there is another connected component, completing the induction. ■