Problems on small world, graph minors, and Laplacians

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1) A 3D grid is the graph consisting of vertices at points (x, y, z), where each of these are integers, say all between 0 and n, and two vertices are connected by an edge if ONE of their coordinates differ by 1 and the others are the same (so there is an edge between $(0, 0, 1)$ and $(0, 1, 0)$). This is the obvious 3D "cube wireframe" analogy to the 2D grids we considered. Suppose we add in one random edge to each vertex v which goes to another vertex u with probability proportional to $d(u, v)^{-q}$. Fix v and let R_d be the ring of vertices a distance at least d from v and at most $2d$ from v. What value should q have so that the probability that the random edge from v jumps into R can be bound by a number independent of d (and hence independent of the choice of ring R_d)? Justify your answer.

2) Go to wikipedia.com and try to find a short path from "Lynah Faithful" to "Zona Zoo" using only links on each page. It may be helpful to know that "Lynah Faithful" refers to hockey fans at Cornell University in Ithaca, NY, and "Zona Zoo" refers to University of Arizona student fans. What is your strategy?

3) Recall the definition of the topological minor relation, that $G_1 \leq_T G_2$ if G_1 can be obtained from G_2 by a sequence of vertex removals, edge removals, and suppressing vertices of degree two. Show that this is equivalent to the relation that a subdivision of G_1 is isomorphic to a subgraph of G_2.

4) A rooted tree (T, v_0) is a tree T together with a vertex $v_0 \in V(T)$. Given a rooted tree, one can define the distance $d(v)$ of any vertex $v \in V(T)$ to v_0 (since trees have unique shortest paths, this is quite robust). This gives a tree order relation, $v \leq w$ if $d(v) \leq d(w)$. Define the relation $(T_1, v_1) \leq (T_2, v_2)$ if there is an isomorphism ϕ of a subdivision of T_1 to a subtree of T_2 (a subtree is a subgraph which is also a tree) which preserves the tree order, i.e., if $v \leq w$ in T_1 then $\phi(v) \leq \phi(w)$. Show this is a quasi-ordering.

5) Show that finite trees are not well-quasi-ordered by the subgraph relation.

6) Use the fact that $L = -QQ^T$ to show that the eigenvalues of L are nonpositive.

7) Compute the Laplacian spectrum of the graph consisting of one cycle of length 4.

8) Let C_6 be the graph consisting of one cycle of length 6. Confirm that $t(C_6) = 6$ by proving (a) T is a spanning tree of C_6 if and only if $T = G - e$ for

some edge, and (b) computing it directly from the Laplacian matrix.

9) For the complete graph K_n, show that the spectrum is $s(K_n) = (0, n, n, n, \ldots, n)$ and that $t(K_n) = n^{n-2}$.