CHAPTER 2: SMOOTH MAPS

DAVID GLICKENSTEIN

1. Introduction

In this chapter we introduce smooth maps between manifolds, and some important concepts.

Definition 1. *A function $f : M \to \mathbb{R}^k$ is a smooth function if for each $p \in M$, there exists a smooth coordinate neighborhood (U, ϕ) of p such that $f \circ \phi^{-1}$ is smooth.*

More generally, we have the following.

Definition 2. *A map $F : M \to N$ is a smooth map if for each point $p \in M$ there exist smooth coordinate neighborhoods (U, ϕ) of p and (V, ψ) of $F(p)$ such that $F(U) \subseteq V$ and*

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$$

is smooth. The map

$$\hat{F} = \psi \circ F \circ \phi^{-1}$$

is called a coordinate representation *of F.*

Basically, we have that F is smooth if and only if it is smooth in its coordinate representations. Note that we have to be a bit careful of the condition $F(U) \subseteq V$.

Consider some examples.

Example 1. *We can use polar coordinates on $\mathbb{R}^2$. The function $f(x, y) = x^2 + y^2$ is equal to $\hat{f}(r, \theta) = r^2$ in a polar coordinate chart (usually one takes a set such as $\left\{(x, y) \in \mathbb{R}^2 : y > 0\right\}$ for the coordinate domain, otherwise one could find that the coordinate chart is not smooth). It follows that f is smooth.*

Lemma 3. *Let $F : M \to N$ be a map. If for each $p \in M$ there exists an open set U such that $F|_U$ is continuous, then F is continuous. If for each $p \in M$ there exists an open set U such that $F|_U$ is smooth, then F is smooth.*

Proof. Suppose that for each $p \in M$ there exists an open set U such that $F|_U$ is continuous. So there is a cover $\{U_p\}_{p \in M}$ with this property. Consider an open set $V \subseteq N$. We note that

$$\left(F|_{U_p}\right)^{-1}(V) = F^{-1}(V) \cap U_p$$

so

$$F^{-1}(V) = \bigcup_p \left(F|_{U_p}\right)^{-1}(V),$$

which is open.

Suppose that for each $p \in M$ there exists an open set U such that $F|_U$ is smooth. Then if $p \in U$ such that $F|_U$ is smooth, there is a coordinate neighborhood $U' \subseteq U$

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of p and $V \subseteq N$ of $F(p)$ such that the coordinate representation is smooth. Since U' is open in M, it follows that F is smooth (note that we used $F(U') \subseteq F(U) \subseteq V$). □

Lemma 4. *Let M, and N be smooth manifolds and let $\{U_\alpha\}$ be an open cover of M. Suppose that for each α we are given a smooth map $F_\alpha : U_\alpha \to N$ such that*

$$F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$$

Then there is a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$.

Proof. We can define $F(p) = F_\alpha(p)$ if $p \in U_\alpha$. By the assumption, this is well-defined, i.e., if $p \in U_\beta$, then $F_\alpha(p) = F_\beta(p)$. Since F is smooth in a neighborhood of each point, it is smooth. Continuous??? □

Lemma 5. *Every smooth map between manifolds is continuous.*

Proof. We know that smooth maps between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ are continuous. Suppose $F : M \to N$ is smooth. Thus for each point $p \in M$ there exist smooth coordinate neighborhoods (U, ϕ) of p and (V, ψ) of $F(p)$ such that $\psi \circ F \circ \phi^{-1}$ is smooth. If $W \subseteq N$ is open, then $W \cap F(M)$ is covered by coordinate charts $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ such at there exist nonempty open coordinate charts (U_α, ϕ_α) such that $F(U_\alpha) \subseteq V_\alpha$ and $\psi_\alpha \circ F \circ \phi_\alpha^{-1}$ is smooth. Since $W \cap V_\alpha$ is open, $\psi_\alpha(W \cap V_\alpha)$ is open, and since $F_\alpha = \psi_\alpha \circ F \circ \phi_\alpha^{-1}$ is smooth, $F_\alpha^{-1}(\psi_\alpha(W \cap V_\alpha))$ is open. However,

$$F_\alpha^{-1}(\psi_\alpha(W \cap V_\alpha)) = \phi_\alpha\left(F^{-1}(W \cap V_\alpha)\right),$$

so $F^{-1}(W \cap V_\alpha)$ is open. Since

$$F^{-1}(W) = \bigcup_{\alpha \in A} F^{-1}(W \cap V_\alpha),$$

we have that $F^{-1}(W)$ is open, and since W is arbitrary it follows that F is continuous. □

Definition 6. *A* diffeomorphism *$M \to N$ between smooth n-dimensional manifolds is a bijective, smooth map with smooth inverse. Two manifolds are* diffeomorphic *if there exists a diffeomorphism between them.*

2. Examples of smooth maps

In this section we present some examples of smooth maps.

Example 2. *Consider the inclusion map $\mathbb{S}^n \to \mathbb{R}^{n+1}$. We need to check the coordinate representations, which are of the form*

$$\left(x^1, \ldots, x^n\right) \to \left(x^1, \ldots, x^n, \pm\sqrt{1 - |x|^2}\right),$$

with domain the open unit disk. Since the coordinate representations are smooth, the function is smooth.

Example 3. *Consider the quotient map $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. The coordinate representations look like*

$$\left(x^0, \ldots, x^n\right) \to \left(\frac{x^1}{x^0}, \ldots, \frac{x^n}{x^0}\right).$$

If we restrict to where $x^0 > 0$, we see that this is a smooth map.

Example 4. *Consider the restriction of the previous map $\mathbb{S}^n \to \mathbb{RP}^n$. The coordinate representation is*

$$\left(x^1, \ldots, x^n\right) \to \left(\frac{x^1}{\sqrt{1-|x|}}, \ldots, \frac{x^n}{\sqrt{1-|x|}}\right)$$

or

$$\left(x^0, \ldots, x^{n-1}\right) \to \left(\frac{x^1}{x^0}, \ldots, \frac{x^{n-1}}{x^0}, \frac{\sqrt{1-|x|}}{x^0}\right).$$

Example 5. *For a product $M \times N$, consider the projection map $\pi : M \times N \to M$. This map is smooth.*

3. Partitions of unity

Partitions of unity are used to glue together two smooth maps in such a way that the new map is smooth. Note that one can easily glue together continuous maps to be continuous, but if applied to smooth maps, the new map is rarely smooth. Consider, for example, the absolute value function.

We need a function that smoothly transitions between the constant function 1 and the constant function 0. To do this, we need a nonzero function with all zero derivatives at a point. Consider the following function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Lemma 7. *The function $f : \mathbb{R} \to \mathbb{R}$ described above is smooth.*

Proof. We need to show that the derivatives all exist at $x = 0$ and are continuous. Suppose we could show that the right-sided derivatives were zero, i.e.,

$$\lim_{x \to 0^+} f^{(k)}(x) = 0$$

for all $k \geq 0$. Since the derivatives from the left are clearly zero, this is sufficient to show that all derivatives are zero.

We have

$$\lim_{x \to 0^+} f^{(0)}(x) = \lim_{x \to 0^+} e^{-1/x} = 0.$$

Now we will prove by induction that for $x > 0$,

$$f^{(k)}(x) = \frac{p_k(x)}{x^{2k}} e^{-1/x}$$

for some polynomial p_k. This is clearly true for $k = 0$. Now supposing the formula for k, we get

$$\begin{aligned} f^{(k+1)}(x) &= \frac{p_k'(x)}{x^{2k}} e^{-1/x} - 2k \frac{p_k(x)}{x^{2k+1}} e^{-1/x} - \frac{p_k(x)}{x^{2k+2}} e^{-1/x} \\ &= \frac{x^2 p_k'(x) - 2kx p_k(x) - p_k(x)}{x^{2(k+1)}} e^{-1/x}, \end{aligned}$$

completing the induction. Now we recall that for any integer k,

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x^k} = 0$$

(this is proved using L'Hospital and induction). It follows that $\lim_{x \to 0^+} f^{(k)}(x) = 0$. □

Using f we can construct smooth cutoff functions.

Lemma 8. *There exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that h is identically zero on $(-\infty, 1]$, h is identically one on $[2, \infty)$, and $0 < h(x) < 1$ for $h \in (1, 2)$.*

Proof. One way to construct such a function is to consider

$$h(x) = \frac{f(2-x)}{f(2-x) + f(x-1)}.$$

Notice that $h(x) \leq 1$ and $h \geq 0$ everywhere. if $2 - x \leq 0$, i.e., $x \geq 2$, then $h(x) = 0$, and if $x - 1 \leq 0$, i.e., $x \leq 1$, then $h(x) = 1$. Note that the denominator is never zero, since $f(2-x)$ is positive if $x < 2$ and $f(x-1)$ is positive if $x > 1$. $\square$

Definition 9. *A smooth function as in the lemma is called a* cutoff function*.*

The other thing we often need is a bump function.

Definition 10. *Let $f : X \to \mathbb{R}$ be a continuous function. The* support *of f, denoted* $\operatorname{supp} f$ *is defined as the closure of the set of points where f is nonzero, i.e.,*

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}.$$

The function f is said to be compactly supported *if the support is a compact set.*

Lemma 11. *There exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $\operatorname{supp} H \subset B(0,2)$ and $H|_{\overline{B(0,1)}} \equiv 1$.*

Proof. We can take $H(x) = h(|x|)$. Clearly this is smooth away from 0 since it is a composition of smooth functions. Notice that $H|_{B(0,1)} \equiv 1$, so it must be smooth at 0 as well. $\square$

Definition 12. *Let X be a topological space and let $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. $\mathcal{S}$ is said to be* locally finite *if each point $x \in X$ has a neighborhood that intersects at most finitely many sets in $\mathcal{S}$.*

Definition 13. *Let M be a topological space and let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an open cover of M. A* partition of unity subordinate to $\mathcal{X}$ *is a collection of continuous functions $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that*

1. $0 \leq \psi_\alpha(x) \leq 1$ *for all* $\alpha \in A$ *and all* $x \in M$.
2. $\operatorname{supp} \psi_\alpha \subset X_\alpha$.
3. *The set of supports* $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$ *is locally finite.*
4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ *for all* $x \in M$.

Note that it is important that the set is locally finite so that the sum makes sense (without any notion of series).

Definition 14. *If ψ_α are all smooth, then we call the collection a* smooth partition of unity*.*

One key fact is that manifolds admit partitions of unity. We have already introduced a number of the concepts, but the problem is ensuring that the supports are locally finite. This will follow from the fact that manifolds are paracompact.

Definition 15. *Let $\mathcal{U}$ be an open cover of a topological space X. A* refinement *is another open cover $\mathcal{V}$ such that for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subseteq U$. The Hausdorff space X is* paracompact *if every open cover has a locally finite refinement.*

The key fact is that paracompact follows from second countable, so every manifold is paracompact.

Lemma 16. *Every topological manifold admits a countable, locally finite cover by precompact open sets.*

Proof. Let M be a manifold. Recall that M admits a countable basis of precompact open sets $\{B_i\}_{i=1}^{\infty}$. We will now construct a cover $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ such that

(1) $\overline{U_i}$ is compact.
(2) $\overline{U_{i-1}} \subseteq U_i$ if $i \geq 2$.
(3) $B_i \subseteq U_i$.

Once we have such a cover, we see that the set $\mathcal{V} = \left\{V_i = U_{i+2} \setminus \overline{U_i}\right\}_{i=1}^{\infty} \cup \{U_1, U_2\}$ is the appropriate cover. Since $\overline{V_i}$ is a closed subset of $\overline{U_{i+2}}$, which is compact, it follows that V_i are precompact. Also, we have that $V_i \cap V_j$ is empty unless $|i-j| \leq 1$, so $\mathcal{V}$ is locally finite. It is clear that

$$\bigcup_{V \in \mathcal{V}} V = \bigcup_{U \in \mathcal{U}} U,$$

so $\mathcal{V}$ is a cover.

To construct $\mathcal{U}$, we proceed inductively. Let $U_1 = B_1$. Then U_1 satisfies all three properties. Inductively, let's suppose we have $U_1, \ldots, U_{k-1}$ ($k \geq 2$) satisfying all three properties. Note that since U_{k-1} is precompact, there is a finite number m_k such that

$$\overline{U_{k-1}} \subset B_1 \cup B_2 \cup \cdots \cup B_{m_k}.$$

Set U_k to be

$$U_k = \bigcup_{i=1}^{\max(m_k, k)} B_i.$$

This will satisfy all the axioms. □

Now a slightly stronger version of paracompactness.

Definition 17. *An open cover $\mathcal{U}$ is* regular *if*

(1) *$\mathcal{U}$ is countable and locally finite.*
(2) *Each $U_i \in \mathcal{U}$ is the domain of a smooth coordinate map $\phi_i : U_i \to \mathbb{R}^n$ whose image is $B(0,3) \subseteq \mathbb{R}^n$.*
(3) *The (countable) collection $\{V_i\}$ still covers M, where $V_i = \phi_i^{-1}(B(0,1))$.*

Lemma 18. *Let M be a manifold. Then every open cover admits a regular refinement. In particular, M is paracompact.*

Proof. Let $\mathcal{X}$ be an open cover of M and let $\{V_j\}$ be a countable, locally finite cover of M by precompact sets guaranteed by Lemma 16. For each $p \in M$, there exists an open neighborhood W_p' of p that intersects only finitely many V_j. Denote by $\mathcal{V}_p$ the set of V_j containing p. Now let

$$W_p = W_p' \cap \left(\bigcap_{V \in \mathcal{V}_p} V \right).$$

Note that this is an open set, since $\mathcal{V}_p$ is a finite set. It follows that if $p \in V_i$, then $W_p \subseteq V_i$. Since $p \in X_p$ for some $X_p \in \mathcal{X}$, for each p we can consider $W_p'' \subseteq W_p \cap X_p$

such that where $\left(W_p'', \phi_p\right)$ is a coordinate ball centered at p, choosing ϕ_p such that $\phi_p : W_p'' \to B(0,3)$.

Now let

$$U_p = \phi_p^{-1}\left(B(0,1)\right).$$

The sets U_p cover M, and the sets W_p'' form a refinement of $\mathcal{X}$ with the correct coordinate map property. We need to restrict to a countable subcover and show that it is locally finite. Define Notice that the set

$$\left\{U_p : p \in \overline{V_i}\right\}$$

forms a cover of $\overline{V_i}$, and since $\overline{V_i}$ is compact, there is a finite subcover $\left\{U_{i,1}, \ldots, U_{i,k(i)}\right\}$. This subcover corresponds to a collection of W_p'' that we denote $\left\{W_{i,1}'', \ldots, W_{i,k(i)}''\right\}$. We can now take the union of these over all $i \in \mathbb{N}$, and call it $\mathcal{U}$. This is countable (since it is the countable union of finite sets), a refinement of $\mathcal{X}$, and has the appropriate coordinate map conditions.

We still need to show that $\mathcal{U}$ is locally finite. Notice that if $W_{i,k}'' \cap W_{i',k'}''$, then since there are j, j' such that $W_{i,k}'' \subseteq V_j$ and $W_{i',k'}'' \subseteq V_{j'}$, we must have $V_j \cap V_{j'}$, $V_i \cap V_j$, and $V_{i'} \cap V_{j'}$ all nonempty. If we fix i, there are only finitely many i' that satisfy this since $\{V_j\}$ is locally finite. It follows that $\mathcal{U}$ is locally finite. (Note: this argument is from the errata on J. Lee's website.) $\square$

Theorem 19. *If M is a smooth manifold and $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ is any open cover of M, there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Proof. Let $\{W_i\}$ be a regular refinement of $\mathcal{X}$, and let $\phi_i : W_i \to B(0,3)$ be the corresponding coordinate maps. By Lemma 11, there is a smooth function H that is 1 on $B(0,1)$ and zero outside $B(0,2)$. For every $i \in \mathbb{N}$, let $f_i : M \to \mathbb{R}$ be the function

$$f_i(p) = \begin{cases} H(\phi_i(p)) & \text{if } p \in W_i \\ 0 & \text{else} \end{cases}$$

These are clearly smooth and equal to one on $U_i = \phi_i^{-1}\left(B(0,1)\right)$.

Now define

$$g_i(p) = \frac{f_i(p)}{\sum_i f_i(p)}.$$

The sum in the bottom makes sense because $\{W_i\}$ is locally finite, so the denominator is a finite sum. Furthermore, since every p is in some U_i, the denominator is never zero. Note that $\sum g_i(p) = 1$ for all $p \in M$. Finally, we need to re-index so that we are indexed by the set A. Every W_i is contained in some X_α, so there is a function $a : \mathbb{N} \to A$ such that $a(i)$ is the appropriate α. It may be that more than one i corresponds to the same α, so we need to sum:

$$\lambda_\alpha(p) = \sum_{i \in a^{-1}(\alpha)} g_i(p)$$

(if $a^{-1}(\alpha) = \varnothing$, then the sum is zero). Again, there can be only finitely many nonzero $g_i(q)$ in a neighborhood of p, so the sum is finite and the function is smooth. It also follows that the supports are locally finite for the same reason, thus $\{\lambda_\alpha\}$ is the partition of unity. $\square$

It will follow that bump functions exist.

Definition 20. *If $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open set containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a bump function for A supported in U if $0 \leq \psi(x) \leq 1$ on M, $\psi \equiv 1$ on A and $\operatorname{supp} \psi \subseteq U$.*

Proposition 21. *Let M be a smooth manifold. For any closed set $A \subseteq M$ and any open set U containing A, there exists a smooth bump function for A supported in U.*

Proof. Consider the cover $\{U, M \setminus A\}$ of M. There is a smooth partition of unity $\{\psi_0, \psi_1\}$ subordinate to this cover. It is clear that $\psi_1 \equiv 0$ on A, so $\psi_0 \equiv 1$ on A since $\psi_0 + \psi_1 \equiv 1$. □

Recall the definition of a smooth function on a closed set A.

Definition 22. *Let $A \subseteq M$ be a closed set. A function $F : A \to N$ is smooth if for every $p \in A$ there is a neighborhood W of p and a smooth function $\tilde{F} : W \to N$ such that $\left.\tilde{F}\right|_{W \cap A} = \left.F\right|_{W \cap A}$.*

Lemma 23. *Let M be a smooth manifold, and let $A \subseteq M$ be a closed subset, and let $f : A \to \mathbb{R}^k$ be a smooth function. For any open set U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\left.\tilde{f}\right|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.*

One might be temped to use the smooth bump function b and look at bf. The problem is that we do not know there is a smooth extenion of f to U, so we cannot get a smooth function this way. The proof is slightly more involved.

Proof. We know that for each p, there exists a neighborhood W_p and function $\tilde{f}_p : W_p \to \mathbb{R}^k$ with the appropriate properties. We can replace W_p with $W_p \cap U$ to ensure $W_p \subseteq U$. The set

$$\{W_p\}_{p \in A} \cup \{M \setminus A\}$$

forms an open cover, and we can find a partition of unity $\{\phi_p\} \cup \{\phi_0\}$ subordinate to it. We then define

$$\tilde{f}(x) = \sum_{p \in M} \phi_p(x) \tilde{f}_p(x).$$

Since $\left\{\operatorname{supp} \tilde{f}_p\right\}_p$ is locally finite, the sum is finite. Since $\operatorname{supp} \tilde{f}_p \subseteq W_p \subseteq U$, we have that $\operatorname{supp} \tilde{f} \subseteq U$ and $\tilde{f}$ is smooth. Finally, for $x \in A$, we have

$$\tilde{f}(x) = \phi_0(x) f(x) + \sum_{p \in M} \phi_p(x) \tilde{f}_p(x) = \left[\phi_0(x) + \sum \phi_p(x)\right] f(x) = f(x).$$

□

Definition 24. *Let M be a topological space. An exhaustion function for M is a continuous function $f : M \to \mathbb{R}$ such that $M_c = \{x \in M : f(x) \leq c\}$ is compact for each $c \in \mathbb{R}$.*

Some popular exhaustion functions are $f(x) = |x|^2$ for $\mathbb{R}^n$ and $f(x) = \frac{1}{1-|x|^2}$ for the open ball of radius 1.

Proposition 25. *Every smooth manifold admits a smooth positive exhaustion function.*

Proof. Let $\{V_j\}_{j\in\mathbb{N}}$ be a countable open cover of a manifold M by precompact sets and let $\{\psi_j\}_{j\in\mathbb{N}}$ be a smooth partition of unity subordinate to $\{V_j\}$. Define $f : M \to \mathbb{R}$ by

$$f(x) = \sum_{j=1}^{\infty} j\psi_j(x).$$

This sum is finite (for each x) since the supports are locally finite, and the function is smooth. It is positive

$$f(x) \geq \sum_{j=1}^{\infty} \psi_j(x) = 1.$$

□