CHAPTER 3: TANGENT SPACE

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1. Tangent space

We shall define the tangent space in several ways. We first try gluing them together. We know vectors in a Euclidean space require a basepoint $x \in U \subset \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$. A C^∞-manifold consists of a number of pieces of $\mathbb{R}^n$ glued together via coordinate charts, so we can define all tangents as follows. Consider what happens during a change of parametrization $\phi : V \to U$. It will take a vector v to $d\phi\,(v)$. This motivates the following:

Definition 1. *$T^{glue}M = \bigsqcup_i (U_i \times \mathbb{R}^n) / \sim$ where for $(x, v) \in U_i \times \mathbb{R}^n$, $(y, w) \in U_j \times \mathbb{R}^n$ we have $(x, v) \sim (y, w)$ if and only iff $y = \phi_j \phi_i^{-1}(x)$ and $w = d\left(\phi_j \phi_i^{-1}\right)_x (v)$.*

The nice thing about this definition is it puts things together and gives the vectors in a good way. We define the tangent space at a point $p \in M$ as $T_p^{\text{glue}}M = \{[p, v] : v \in \mathbb{R}^n\}$. It is easy to see that $T_p^{\text{glue}}M$ is an n-dimensional vector space. It is also easy to see that there is a map $\pi : T^{\text{glue}}M \to M$ defined by $\pi([p, v]) = p$ (since the parts of M are really equivalence classes modulo equivalence. It also makes it clear that $T^{\text{glue}}M$ is a C^∞ manifold.

We can define tangent spaces in two other ways.

Definition 2. *$T_p^{path}M = \{paths\ \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ where $\alpha \sim \beta$ if $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ for every i such that $p \in U_i$. $T^{path}M = \bigsqcup_{p \in M} T_p^{path}M$.*

This is a more geometric definition. Note that there is a map $\pi : T^{\text{path}}M \to M$ defined by $\pi(\gamma) = \gamma(0)$.

We shall show that $T^{\text{path}}M$ and $T^{\text{glue}}M$ are equivalent. The maps are

$$\Phi : T_p^{\text{path}}M \to T_p^{\text{glue}}M$$

defined by

$$\Phi([\gamma]) = \left[\phi_i \circ \gamma(0), (\phi_i \circ \gamma)'(0)\right].$$

The inverse map is

$$\Psi : T^{\text{glue}}M \to T^{\text{path}}M$$

defined by

$$\Psi([\phi_i(p), v]) = \left[t \to \phi_i^{-1}(\phi_i(p) + tv)\right].$$

It is clear that if well defined, they are inverses of each other. We need to show that Φ and Ψ are well-defined. Clearly Φ is well defined because $\phi_i \circ \gamma(0) =$

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$\phi_i \circ \beta(0), (\phi_i \circ \gamma)'(0) = (\phi_i \circ \beta)'(0)$ for any $\beta \in [\gamma]$. Also for any $(\phi_j(p), w) \in [\phi_i(p), v]$ must satisfy $d\left(\phi_i \circ \phi_j^{-1}\right)_{\phi_j(p)} v = w$. Notice that

$$\left\{\phi_j \phi_i^{-1}(\phi_i(p) + tv)\right\}'(0) = d\left(\phi_j \circ \phi_i^{-1}\right)_{\phi_i(p)} v = w = (\phi_j(p) + tw)'(0).$$

The third way is in terms of germs of functions. A germ of a function is an equivalence class of functions.

Definition 3. *$Germs_p$ is the set of functions $f \in C^\infty(U_f)$ for $p \in U_f \subset M$ modulo the equivalence that $[f] = [g]$ iff $f(x) = g(x)$ for all $x \in U_f \cap U_g$. Note that $Germs_p$ are an algebra since $[f] + [g] = [f + g]$ is well-defined, etc.*

Definition 4. *A derivation of germs is an $\mathbb{R}$-linear map X :$Germs_p \to \mathbb{R}$ which satisfies*

$$X(fg) = f(p) X(g) + X(f) g(p).$$

Definition 5. *We define $T_p^{der} M$ to be the set of derivations of germs at p.*

Proposition 6. *Alternately, we may define the $T_p^{der} M$ to be the set of derivations of smooth functions at p.*

Proof. Suppose $X : C^\infty(M) \to \mathbb{R}$ is a derivation at p. Then it determines a derivation of germs in the obvious way. Conversely, suppose $[f]$ is a germ at p. Then there is a representative $f : U \to \mathbb{R}$, and within that open set is a coordinate ball B centered at p. Taking a smaller ball, we have a compact (closed) coordinate ball B' around p within the domain U of f. We can consider the function $x \to b(x) f(x)$, where b is a smooth bump function supported in U that is one on the ball B'. These □

This definition is nice because it shows how tangent vectors act on functions. We note derivations are a vector space since

$$\begin{aligned}(X+Y)(fg) &= X(f) g(p) + f(p) X(g) + Y(f) g(p) + f(p) Y(g) \\ &= (X+Y)(f) g(p) + f(p)(X+Y)(g).\end{aligned}$$

A good example of a germ on $U \subset \mathbb{R}^n$ is $\left.\frac{\partial}{\partial x^i}\right|_p$ since

$$\left.\frac{\partial}{\partial x^i}\right|_p (fg) = \frac{\partial f}{\partial x^i}(p) g(p) + f(p) \frac{\partial g}{\partial x^i}(p).$$

These are linearly independent since $\left.\frac{\partial}{\partial x^i}\right|_p x^j = I_i^j$. We see that

$$X(1) = 1 \cdot X(1) + X(1) \cdot 1$$

so $X(1) = 0$. Similarly,

$$X\left(\left(x^i - p^i\right)\left(x^j - p^j\right)\right) = 0.$$

So by Taylor series:

$$f(x) = f(p) + \left.\frac{\partial f}{\partial x^i}\right|_p \left(x^i - p^i\right) + O\left(|x-p|^2\right).$$

We have formally that $\frac{\partial}{\partial x^i}\big|_p$ span $T_p^{\mathrm{der}}U$. To make this argument rigorous, we know that

$$\begin{aligned} f(x) &= f(p) + \int_0^1 \frac{d}{dt} f(tx + (1-t)p)\, dt \\ &= f(p) + \int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tx+(1-t)p} \left(x^i - p^i\right) dt. \end{aligned}$$

Hence if we apply a derivation X we have

$$\begin{aligned} X(f) &= \int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_p dt \cdot X\left(x^i - p^i\right) + X\left(\int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tx+(1-t)p} dt\right) \cdot \left(p^i - p^i\right) \\ &= \left.\frac{\partial f}{\partial x^i}\right|_p \cdot X\left(x^i - p^i\right). \end{aligned}$$

Hence for $U \subset \mathbb{R}^n$ we have a correspondence

$$T_p^{\mathrm{der}}U \to \mathbb{R}^n$$

given by

$$X \to \left(X\left(x^1 - p^1\right), \ldots, X\left(x^n - p^n\right)\right)$$

which is an invertible linear map with inverse

$$\begin{aligned} \mathbb{R}^n &\to T_p^{\mathrm{der}}U \\ \left(s^1, \ldots, s^n\right) &\to \left(X(f) = \left.\frac{\partial f}{\partial x^i}\right|_p s^i\right). \end{aligned}$$

On a manifold, we define

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\phi_i(p)} (f \circ \phi_i)$$

for coordinates $\left(x^1, \ldots, x^n\right) = \phi_i(p)$. Notice that under a change of coordinates from $\left(y^1, \ldots, y^n\right) = \phi_j(p)$ we have that

$$\begin{aligned} \frac{\partial}{\partial x^k} &= \left.\frac{\partial}{\partial x^k}\right|_{\phi_i(p)} (f \circ \phi_i) \\ &= \left.\frac{\partial}{\partial x^k}\right|_{\phi_j \circ \phi_i^{-1} \circ \phi_i(p)} \left(f \circ \phi_j \circ \phi_i^{-1} \circ \phi_i\right) \\ &= \left.\frac{\partial y^\ell}{\partial x^k}\right|_{\phi_i(p)} \left.\frac{\partial}{\partial y^\ell}\right|_{\phi_j(p)} (f \circ \phi_j) \end{aligned}$$

Also, we have the projection $\pi : T^{\mathrm{der}}M \to M$.

Proposition 7. *Let $M = \mathbb{R}^n$. The derivations $\frac{\partial}{\partial x^i}\big|_p$ form a basis for the derivations at p.*

Proof. We first see that $X(c)=0$ if c is a constant function. By linearity of the derivation, we need only show that $X(1)=0$. We compute:

$$\begin{aligned} X(1) &= X(1\cdot 1) \\ &= 1\cdot X(1) + X(1)\cdot 1 \\ &= 2X(1). \end{aligned}$$

We conclude that $X(1)=0$.

Now, let X be a derivation and f a smooth function. We can write f as

$$\begin{aligned} f(x) &= f(p) + \int_0^1 \frac{d}{dt} f(tx+(1-t)p)\,dt \\ &= f(p) + \int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tx+(1-t)p} \left(x^i - p^i\right) dt. \end{aligned}$$

By linearity and the derivation property, we have

$$\begin{aligned} X(f) &= X(f(p)) + X\left(\int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tx+(1-t)p} \left(x^i-p^i\right) dt\right) \\ &= 0 + \left(\int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tp+(1-t)p} dt\right) X\left(x^i-p^i\right) + X\left(\int_0^1 \left.\frac{\partial f}{\partial x^i}\right|_{tx+(1-t)p} dt\right)\left(p^i-p^i\right) \\ &= \left.\frac{\partial f}{\partial x^i}\right|_p X\left(x^i-p^i\right). \end{aligned}$$

So, $X\left(x^i-p^i\right)$ are just some numbers, and so we see that X is a linear combination of $\left.\frac{\partial}{\partial x^i}\right|_p$, meaning that these span the space of derivations! Since it is clear that $\left.\frac{\partial}{\partial x^i}\right|_p$ and $\left.\frac{\partial}{\partial x^j}\right|_p$ are linearly independent for each $i\neq j$ (consider the functions x^i-p^i), the result follows. □

Definition 8. *Given any smooth map $F: M\to N$, there is a push forward $F_*: T_pM \to T_{F(p)}M$ given as follows:*

$$\begin{aligned} F_*^{path}[\gamma] &= [F\circ\gamma] \\ \left(F_*^{der}X\right) f &= X(f\circ F). \end{aligned}$$

Definition 9. *In any coordinate neighborhood (U,ϕ) of p, we define the derivation $\left.\frac{\partial}{\partial x^k}\right|_p$ by*

$$\left.\frac{\partial}{\partial x^k}\right|_p = \phi_*^{-1} \left.\frac{\partial}{\partial x^k}\right|_{\phi(p)}$$

We may now see that $T_p^{\mathrm{der}}M$ is isomorphic to $T_p^{\mathrm{path}}M$. The map is

$$[\gamma] \to \left\{ f \to \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \right\}.$$

We note that

$$\left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{\partial\left(f\circ\phi_i^{-1}\right)}{\partial x^j}\right|_{\phi_i\circ\gamma(0)} \cdot \left.\frac{d\left(\phi_i\circ\gamma\right)^j}{dt}\right|_0$$

and hence it is well-defined up to equivalence of paths. Note that $\left\{\phi_i^{-1}\left(p+te_k\right)\right\}_{k=1}^n$ form a basis for $[\gamma]$ and map to $\left.\frac{\partial}{\partial x^k}\right|_p$ so this is a linear isometry.

We will now use whichever definition we wish. Also note the following:

Proposition 10. *If $p \in U \subseteq M$ is an open set, then*

$$T_pM \cong T_pU.$$

Therefore, we will not make a distinction.

2. Computation in coordinates

Let's compute the push-forward in coordinates. Recall that $\left\{\left.\frac{\partial}{\partial x^k}\right|_p\right\}_{k=1}^m$ is a basis for T_pM. Now, suppose that $\left\{\left.\frac{\partial}{\partial y^a}\right|_{F(p)}\right\}_{a=1}^n$ is a basis for $T_{F(p)}N$. Given a smooth map $F : M \to N$, we should be able to compute the push forward in coordinates. If $X \in T_pM$, we can write it in terms of the basis,

$$X = X^k \left.\frac{\partial}{\partial x^k}\right|_p$$

for some numbers $X^k \in \mathbb{R}$. To compute the push forward, which is a linear map, we have that

$$F_*X = X^k F_* \left.\frac{\partial}{\partial x^k}\right|_p .$$

First, let's suppose $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. To compute $F_* \left.\frac{\partial}{\partial x^k}\right|_p$, for $f \in C^\infty(N)$ we need to compute

$$\begin{aligned}\left(F_* \left.\frac{\partial}{\partial x^k}\right|_p\right) f &= \left.\frac{\partial}{\partial x^k}\right|_p (f \circ F) \\ &= \left.\frac{\partial f}{\partial y^a}\right|_{F(p)} \left.\frac{\partial y^a}{\partial x^k}\right|_p\end{aligned}$$

(note the summation) where, in the second expression, we really mean

$$\left.\frac{\partial y^a}{\partial x^k}\right|_p = \left.\frac{\partial y^a\left(F(x)\right)}{\partial x^k}\right|_p = \left.\frac{\partial F^a}{\partial x^k}\right|_p$$

if $F = \left(F^1, \ldots, F^n\right)$ is written in y-coordinates. Notice that once we have specified the coordinates, we have an expression for F_* in terms of the differential.

Now suppose we are on a manifold, then

$$\left(F_* \left.\frac{\partial}{\partial x^k}\right|_p\right) = \left(\psi_*^{-1}\left(\psi_* F_* \phi_*^{-1}\right)\phi_* \left.\frac{\partial}{\partial x^k}\right|_\phi\right).$$

The middle map is known to us, as it is the differential of a map between $\mathbb{R}^m$ and $\mathbb{R}^n$, that is

$$\psi_* F_* \phi_*^{-1} = \left(\frac{\partial \hat{F}^a}{\partial x^k}\left(\phi\left(p\right)\right)\right)_{\substack{k=1,\ldots,m \\ a=1,\ldots,n}}$$

where $\hat{F} = \psi \circ F \circ \phi^{-1}$. In particular, we get

$$\left(F_* \left. \frac{\partial}{\partial x^k} \right|_p \right) = \frac{\partial \hat{F}^a}{\partial x^k} \left(\phi \left(p \right) \right) \left. \frac{\partial}{\partial y^a} \right|_{F(p)}$$

One can also consider change of coordinates. If (U, ϕ) and (V, ψ) are coordinate charts with coordinates $\left(x^i\right)$ and $\left(\tilde{x}^i\right)$, then any tangent vector can be written as

$$X = X^i \left. \frac{\partial}{\partial x^i} \right|_p = \tilde{X}^i \left. \frac{\partial}{\partial \tilde{x}^i} \right|_p .$$

How are X^i and $\tilde{X}^i$ related? We can compute:

$$\begin{aligned} \tilde{X}^i \left. \frac{\partial}{\partial \tilde{x}^i} \right|_p &= \tilde{X}^i \psi_*^{-1} \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{\psi(p)} \\ &= \tilde{X}^i \phi_*^{-1} \phi_* \psi_*^{-1} \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{\psi(p)} \\ &= \tilde{X}^i \phi_*^{-1} \left(\phi \circ \psi^{-1} \right)_* \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{\psi(p)} \\ &= \tilde{X}^i \phi_*^{-1} \left[\frac{\partial \left(\phi \circ \psi^{-1} \right)^k}{\partial \tilde{x}^i} \left(\psi \left(p \right) \right) \left. \frac{\partial}{\partial x^k} \right|_{\phi(p)} \right] \\ &= \tilde{X}^i \frac{\partial \left(\phi \circ \psi^{-1} \right)^k}{\partial \tilde{x}^i} \left(\psi \left(p \right) \right) \left. \frac{\partial}{\partial x^k} \right|_p \end{aligned}$$

and so

$$X^k = \tilde{X}^i \frac{\partial \left(\phi \circ \psi^{-1} \right)^k}{\partial \tilde{x}^i} \left(\psi \left(p \right) \right).$$

Example 1. *Calculate the differential of the map* $F : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{CP}^1$*.*