COVERING SPACES

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1. Introduction and Examples

We have already seen a prime example of a covering space when we looked at the exponential map $t \to \exp(2\pi i t)$, which is a map $\mathbb{R} \to S^1$. The key property is tied up in this definition.

Definition 1. *A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ such that there exists and open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α. (Draw picture of maps).*

Remark 1. *Sometimes surjectivity is required, which prevents $p^{-1}(U_\alpha)$ to be non-empty. Hatcher does not make this requirement.*

Let's look at some examples:

- We already looked at the exponential map, which can be visualized as the map of a helix to a circle.
- There are other coverings of the circle, say by $z \to z^n$ in the complex plane. These are all maps $S^1 \to S^1$.
- Lots of coverings of $S^1 \vee S^1$. Try all covers with two preimages and also simply connected cover. See page 58 in Hatcher.

Remark 2. *Usually we will study path connected covers, otherwise we can always make lots of disjoint copies of the covers.*

We will classify all covers of most spaces in terms of the fundamental group.

2. Lifting

The main properties we will need are lifting properties. We have already seen these in the case of the exponential map.

Definition 2. *Given a covering $p : \tilde{X} \to X$, a lifting (or lift) of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$.*

Proposition 3 (Homotopy lifting property)**.** *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a lifting $\tilde{f}_0 : Y \to \tilde{X}$ of f_0, there is a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ that lifts f_t.*

Proof. The proof is exactly the same as that for $\mathbb{R} \to S^1$. □

Remark 3. *Sometimes called the* covering homotopy property*.*

Date: February 24, 2011.

As we have seen , restricting Y to be a point gives a path lifting property for covering spaces, so for any path $f : I \to X$, given a lift of $f(0)$, i.e., a point in $\tilde{X}$ that projects via p to $f(0)$, the path can be uniquely lifted to a path in $\tilde{X}$. Note that even if f is a loop, the lifting $\tilde{f}$ may not be a loop. This is extremely important.

Proposition 4. *The induced map $p_* : \pi_1\left(\tilde{X}, \tilde{x}_0\right) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ consists of homotopy classes of loops in X based at x_0 that lift to loops in $\tilde{X}$ based at $\tilde{x}_0$.*

Proof. Consider an element in the kernel represented by a loop $\tilde{f} : I \to \tilde{X}$. That means there is a homotopy of $f = p \circ \tilde{f}$ to the constant loop. Note that, by definition, $\tilde{f}$ is a lift of f, and so we can lift the homotopy f_t by the homotopy lifting property to a homotopy $\tilde{f}_t$. Since f_1 is the constant loop, $\tilde{f}_1$ must be the constant loop, and so we get that $\tilde{f}$ is homotopic to a constant loop and p_* is injective.

For the second statement, clearly such loops must be in $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$. Conversely, any loop in the image is homotopic to the image of a loop in $\tilde{X}$. □

Note that if $p : \tilde{X} \to X$ is a covering map, then the cardinality of $p^{-1}(x)$ is locally constant, and if X is connected, then then the cardinality is the same for every $x \in X$. If the cardinality is finite, say n, then we say p is an n-sheeted covering or that n is the number of sheets.

Proposition 5. *The number of sheets of a covering space $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ with both X and $\tilde{X}$ path connected equals the index $\left[\pi_1(X, x_0) : p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)\right]$ (the index of $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ in $\pi_1(X, x_0)$).*

Proof. For any loop $g \in X$ based at x_0, let $\tilde{g}$ denote its (unique!) lift starting at $\tilde{x}_0$. Given $[h] \in H = p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$, we have that $\tilde{h}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$. Thus $\left(\tilde{h} \cdot \tilde{g}\right)(1) = \tilde{g}(1)$ for any loops g and h such that $[h] \in H$. It follows that there is a well-defined map $\Phi : H \backslash \pi_1\left(\tilde{X}, \tilde{x}_0\right) \to p^{-1}(x_0)$ given by $\Phi(H[g]) = \tilde{g}(1)$. We need to show that Φ is a bijection.

Since $\tilde{X}$ is path connected, for any $\tilde{x} \in p^{-1}(x_0)$, there is a path $\tilde{g}$ from $\tilde{x}_0$ to $\tilde{x}$, and it must project to a loop g in X based at x_0. Thus Φ is surjective.

Suppose $\Phi(H[g]) = \Phi(H[g'])$. Then $\tilde{g}(1) = \tilde{g}'(1)$, and so we have that $g \cdot \bar{g}'$ lifts to a loop in $\tilde{X}$, i.e., $[g][g']^{-1} \in H$, and so $H[g] = H[g']$. □

We can use path lifting to produce liftings of other maps (not just paths and homotopies). First recall the following definition.

Definition 6. *A space X is* locally path connected *if for every point $x \in X$ contained in an open set U, there is an open set $V \subseteq U$ containing x that is path connected.*

Proposition 7 (Lifting criterion)**.** *Let $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ be a covering space and $f : (Y, y_0) \to (X, x_0)$ be a map, with Y path connected and locally path connected. Then a lift*

$$\tilde{f} : (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right)$$

of f exists if and only if $f_*\left(\pi_1\left(Y,y_0\right)\right) \subseteq p_*\left(\pi_1\left(\tilde{X},\tilde{x}_0\right)\right)$.

Proof. Clearly if a lift $\tilde{f}$ exists, then $p_*\tilde{f}_* = f_*$, so

$$f_*\left(\pi_1\left(Y,y_0\right)\right) = p_*\tilde{f}_*\left(\pi_1\left(Y,y_0\right)\right)$$

and the inclusion is clear.

Suppose $f_*\left(\pi_1\left(Y,y_0\right)\right) \subseteq p_*\left(\pi_1\left(\tilde{X},\tilde{x}_0\right)\right)$. We wish to define the lift $\tilde{f}$ by lifting paths. We need $\tilde{f}\left(y_0\right) = \tilde{x}_0 \in p^{-1}\left(x_0\right)$. Now consider $y \in Y$. Since y is path connected, there is a path γ in Y from y_0 to y. This gives a path $f\circ\gamma$ in X that can be lifted to a unique path $\widetilde{f\circ\gamma}$ starting at $\tilde{x}_0$. We define $\tilde{f}\left(y\right) = \widetilde{f\circ\gamma}\left(1\right)$. Certainly $p\circ\tilde{f} = f$, but we need to make sure that $\tilde{f}$ is well-defined and continuous.

Suppose γ' is another path from y_0 to y. Then there is a lifted path $\widetilde{f\circ\gamma'}$ that could have been used instead. Then we have a loop $\bar{\gamma}\cdot\gamma'$ in Y based at y_0, and we know by assumption that

$$f_*\left[\bar{\gamma}\cdot\gamma'\right] = p_*\left[\alpha\right]$$

for some loop α in $\tilde{X}$ based at $\tilde{x}_0$, i.e., there is a loop $p\circ\alpha$ homotopic to $f\circ\left(\bar{\gamma}\cdot\gamma'\right)$. This homotopy can be lifted to a homotopy between $\widetilde{f\circ\left(\bar{\gamma}\cdot\gamma'\right)}$ and $\widetilde{p\circ\alpha} = \alpha$ (by uniqueness of lifting). Since α is a loop, so must $\widetilde{f\circ\left(\bar{\gamma}\cdot\gamma'\right)}$. It follows that $\widetilde{f\circ\gamma'}\left(1\right) = \widetilde{f\circ\gamma}\left(1\right)$ (again by uniqueness of lifting).

To show that the lift $\tilde{f}$ is continuous, we will show it is continuous at each point y, i.e., for every y and open set $W \subseteq \tilde{X}$ containing $f\left(y\right)$, there is an open set $V \subseteq Y$ containing y such that $f\left(V\right) \subseteq W$. Consider $y \in Y$ and let $\tilde{U}$ be a neighborhood of $\tilde{f}\left(y\right)$ such that $p : \tilde{U} \to U$ is a homeomorphism (for some neighborhood U of $f\left(y\right)$). [Note: given any open set W, there is a subset like $\tilde{U}$ since $\tilde{X}$ is a covering space]. Now, consider $\left(f\circ p^{-1}\right)^{-1}\tilde{U}$, which is an open set in Y containing y. Since Y is locally path connected, there is a path connected open set $V \subseteq \left(f\circ p^{-1}\right)^{-1}\tilde{U}$ containing y, and so for any $y' \in V$, there is a path from y_0 to y' that goes through y. Thus $\tilde{f}\left(V\right)$ gets mapped into $\tilde{U}$ by the uniqueness of path lifting. □

Proposition 8 (Unique lifting property). *Given a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$ with two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ that agree at one point in Y, then if Y is connected, these two lifts agree on all of Y.*

Proof. We will consider the set S of points where $\tilde{f}_1$ and $\tilde{f}_2$ agree, which is nonempty by assumption. Let $y \in Y$ and let U be an evenly covered neighborhood of $f\left(y\right)$. If $y \notin S$, then $\tilde{f}_1\left(y\right) \neq \tilde{f}_2\left(y\right)$ and so $\tilde{f}_1\left(y\right) \in \tilde{U}_1$ and $\tilde{f}_2\left(y\right) \in \tilde{U}_2$ such that both open sets are homeomorphic to U and the two sets are disjoint. It follows that $\tilde{f}_1^{-1}\left(\tilde{U}_1\right) \cap \tilde{f}_2^{-1}\left(\tilde{U}_2\right)$ is an open neighborhood of y not in S, and so S is closed. Similarly, if $y \in S$, then $\tilde{f}_1\left(y\right) = \tilde{f}_2\left(y\right) \in \tilde{U}_1$. Since the preimage of an open set is open, $\tilde{f}_1^{-1}\left(\tilde{U}_1\right) \cap \tilde{f}_2^{-1}\left(\tilde{U}_1\right)$ must map to the single sheet, and so the two images must be equal and there is an open neighborhood of y in S, so S is open. Since S is open, closed, and nonempty and Y is connected, $S = Y$. □

3. Classification of covering spaces

The main things we will prove are the following:

Theorem 9. *If a space Y is path connected and locally path connected, then Y has a simply connected covering space if and only if Y is semilocally simply connected.*

Theorem 10. *Let X be path connected, locally path connected, and semilocally simply connected. Then there is a bijection between the set of basepoint preserving isomorphism classes of path connected covering spaces*

$$p : (X, \tilde{x}_0) \to (X, x_0)$$

and the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ to the covering space $\left(\tilde{X}, \tilde{x}_0\right)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1\left(\tilde{X}, \tilde{x}_0\right)$.

Definition 11. *A space X is* semilocally simply connected *if each point $x \in X$ has a neighborhood U such that $\iota_* \pi_1(U, x) \subseteq \pi_1(X, x)$ is trivial.*

Remark 4. *Another way to express this is that for the neighborhood U, any loop based at x in U is homotopic to the trivial loop, but that homotopy may take the loop outside U. If the homotopy were required to stay in U, the condition would be called* locally simply connected.

It is clear that if a space X admits a simply connected covering space $\tilde{X}$, then it must be semilocally simply connected, since given any point $x \in X$, and a loop γ based at x, we can let U be an evenly covered neighborhood of X. The loop can be lifted to a path $\tilde{\gamma}$ in the universal cover $\tilde{X}$ and since $\gamma \subseteq U$ and U is evenly covered, it follows that $\tilde{\gamma}$ is in only one sheet, and thus is a loop. The loop $\tilde{\gamma}$ is homotopic to the trivial loop since $\tilde{X}$ is simply connected. The projection of the homotopy will become a homotopy of γ to the trivial loop. Thus X is semilocally simply connected.

3.1. **Simply connected covering space.** In this section we will show that if X is path connected, locally path connected, and semilocally simply connected, then X has a simply connected covering space, called the *universal covering space* (for reasons we will see later).

Note that if $\tilde{X}$ is a simply connected covering space, then given a point $\tilde{x}_0 \in \tilde{X}$, one can identify points $\tilde{x} \in \tilde{X}$ with homotopy classes of paths $[\tilde{\gamma}]$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tilde{x}$. By path and homotopy lifting, every path in X starting at $x_0 = p(\tilde{x}_0)$ lifts to a path in $\tilde{X}$ starting at $\tilde{x}_0$, as do homotopies. Thus homotopy classes of paths in X correspond to points in $\tilde{X}$ (by unique path lifting and path connectedness, there is a path in X corresponding to each point in $\tilde{X}$).

We will define the universal covering space in exactly this way:

$$\tilde{X} = \{[\gamma] : \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0\}.$$

The map is

$$p([\gamma]) = \gamma(1),$$

which is well-defined since homotopies fix endpoints. Now, we need to give $\tilde{X}$ a topology that makes p into a covering map (so we need to show that every point in X has an evenly covered neighborhood, and that p is continuous). Finally, we need to show that $\tilde{X}$ is simply connected.

We will give $\tilde{X}$ a topology by defining a neighborhood basis. Recall that a neighborhood basis is a collection of sets such that every point is in one of the sets, and for any point that is in the intersection of two such sets, there is a third set in the collection containing the point that is contained in the intersection. We will define neighborhoods of each point as follows. Let $\mathcal{U}$ be a collection of path connected open sets that cover X (these exist because X is locally path connected). Now define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] : \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

Notice that $U_{[\gamma]}$ depends only on the homotopy class of γ in X (also note that the homotopy class is that homotopy class in X). Notice that $p : U_{[\gamma]} \to U$ is surjective since U is path connected.

Important observation: if $\iota_* p_* : \pi_1\left(U_{[\gamma]}\right) \to \pi_1(U) \to \pi_1(X)$ is trivial (if it is true for one basepoint, it is true for all since U is path connected), then p is injective, since different choices of η are homotopic in X. If X is semilocally simply connected, then we may choose the collection $\mathcal{U}$ so that each $U \in \mathcal{U}$ has the property that the second map in the composition is trivial, so the composition is trivial, too. Thus $p : U_{[\gamma]} \to U$ is a bijection.

Claim 1. *$U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$.*

Proof. If $[\gamma'] \in U_{[\gamma]}$, then $\gamma' \simeq \gamma \cdot \eta$ in X. So all elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ for some path μ in U. But then $\eta \cdot \mu$ is an appropriate path in U, and so $U_{[\gamma']} \subseteq U_{[\gamma]}$. The other inclusion follows similarly. □

In particular, we have that if $[\alpha] \in U_{[\gamma]} \cap U_{[\gamma']}$, then $U_{[\gamma]} = U_{[\alpha]} = U_{[\gamma']}$. Now suppose $[\alpha] \in U_{[\gamma]} \cap V_{[\gamma']}$. Then $U_{[\gamma]} = U_{[\alpha]}$ and $V_{[\gamma']} = V_{[\alpha]}$. If $W \subseteq U \cap V$, and $W \subseteq \mathcal{U}$ and $\alpha(1) \in W$, then $W_{[\alpha]} \subseteq U_{[\gamma]} \cap V_{[\gamma']}$. Since every $[\gamma] \in \tilde{X}$ is contained in $U_{[\gamma]}$, it follows that $\left\{U_{[\gamma]}\right\}_{[\gamma]}$ form a basis for a topology. So we need to choose $\mathcal{U}$ so that each set is path connected, pushes forward trivially in the fundamental group, and is a basis for the topology of X. This can be done as follows by letting $\mathcal{U}$ be the set of all path connected open sets U such that $\pi_1(U) \to \pi_1(X)$ is trivial. Then if $U \cap V \neq \varnothing$, then there is a path connected open set contained in the intersection (around any point) since X is locally path connected, and using inclusion, it must still satisfy that the fundamental group maps trivially into the fundamental group of X.

We can now see that $p : U_{[\gamma]} \to U$ is a homeomorphism. The map p is continuous since for any point $p([\alpha]) = \alpha(1) \in V \subseteq U$, where V is open, we have that $p(V_{[\alpha]}) = V$. It is open since any point $[\alpha] \in U_{[\gamma]}$ has an open neighborhood $V_{[\gamma']}$ whose image is $V \in \mathcal{U}$, and hence open.

The same argument shows that $p : \tilde{X} \to X$ is continuous. It is a covering space: Given any $x \in X$, x is contained in some $U \in \mathcal{U}$. We may then consider the sets $U_{[\gamma]}$ for all homotopy classes $[\gamma]$ of paths from x_0 to x. Each set $U_{[\gamma]}$ is homeomorphic to U via p. For any two classes $[\gamma]$, $[\gamma']$ if $U_{[\gamma]} \cap U_{[\gamma']} \neq \varnothing$ then the two sets are equal. Thus we find that $p^{-1}(U)$ is a disjoint collection of sets homeomorphic to U via p.

Claim 2. *$\tilde{X}$ is simply-connected.*

Proof. Given $x_0 \in X$, there is a natural basepoint for $\tilde{X}$ given by the homotopy class of the constant loop $[x_0]$. We first show that $\tilde{X}$ is path connected: given any point $[\gamma] \in \tilde{X}$, there is the path

$$t \to \left[\gamma|_{[0,t]}\right] = [s \to \gamma(st)].$$

in $\tilde{X}$ that connects $[x_0]$ to $[\gamma]$.

Let's take the basepoint of $\tilde{X}$ to be a constant loop $[x_0]$. Let $\tilde{\alpha}(s)$ be a loop in $\tilde{X}$ based at $[x_0]$. Then $\gamma(s) = p \circ \tilde{\alpha}(s)$ is a loop in X based at x_0.

Claim: The path $\tilde{\beta}(t) = [s \to \gamma(st)]$ is a lift of γ such that $\tilde{\beta}(0) = [x_0]$. This follows because $p\left(\tilde{\beta}(t)\right) = \gamma(t)$, so it is a lift.

It follows by uniqueness of lifts that $\tilde{\beta}(t) = \tilde{\alpha}(t)$. Since $\tilde{\alpha}$ is a loop, we must have that $\tilde{\beta}(0) = \tilde{\beta}(1)$, which means that $[x_0] = [\gamma]$. Hence γ is null-homotopic. Since p_* is injective, this implies that $\tilde{\alpha}$ is null homotopic, and hence $\tilde{X}$ is simply connected. □

Proposition 12. *If $\tilde{X}_1 \to X$ is a covering space and $\tilde{X} \to X$ is a simply connected covering space, then $\tilde{X}$ is a covering space of $\tilde{X}_1$. Thus there is a partial ordering of covering spaces.*

Proof. Since $\pi_1\left(\tilde{X}\right)$ is trivial, the lifting criterion says that the map $\tilde{X} \to X$ can be lifted to $\tilde{X} \to \tilde{X}_1$. The fact that this is a covering can be checked. □

For this reason, a simply connected covering space is called a *universal cover.* In fact, it is unique up to isomorphism.

Definition 13. *An* isomorphism *between covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ f = p_1$.*

Proposition 14. *If $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are covering spaces and $\tilde{X}_1$ and $\tilde{X}_2$ are simply connected, then the covering spaces are isomorphic.*

Proof. As in the previous proposition, the projection maps can be lifted. If we are careful with the points, we can ensure that the two lifts are inverses of each other. □

3.2. **Subgroups of π_1.** In fact, our understanding of simply connected coverings in terms of paths and the existence of simply connected coverings gives a more general result.

Proposition 15. *Suppose X is path connected, locally path connected, and semilocally simply connected. Then for every subgroup $H < \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in X_H$.*

Proof. Let $\tilde{X}$ be the universal cover. We know that points in $\tilde{X}$ correspond to homotopy classes of paths in X. We now define X_H as the quotient of $\tilde{X}$ by the equivalence relation

$$[\gamma] \sim [\gamma'] \text{ iff } \gamma(1) = \gamma'(1) \text{ and } \left[\gamma \cdot \overline{\gamma'}\right] \in H.$$

(It is an equivalence relation since H is a group!) Notice that for the neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$, if $[\gamma] \sim [\gamma']$ then the whole neighborhoods are identified since $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$. It follows that X_H is a covering space.

Choose $\tilde{x}_0$ to be the equivalence class of the constant loop at x_0. Clearly, if $[\gamma] \in H$, then γ lifts to the loop $t \to \left[\gamma|_{[0,t]}\right]$ in X_H (since $[\gamma] \sim [x_0]$). Similarly, if α is a loop in X_H based at $\tilde{x}_0$, then $(p \circ \alpha)(t)$ is a loop in Z and $\alpha(t) = \left[p \circ \alpha|_{[0,t]}\right]$. Since α is a loop, we have that $[p \circ \alpha] \in H$. Thus $p_* (\pi_1 (X_H, \tilde{x}_0)) = H$. □

This completes the proof of the first part of the classification theorem (since every covering space induces a subgroup of $\pi_1 (X, x_0)$ by pushing froward the fundamental group by p. For the second statement, we need to understand the correspondence between isomorphism classes of covering spaces and the subgroups they generate.

Proposition 16. *Two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism ϕ such that $\phi(\tilde{x}_1) = \tilde{x}_2$ if and only if*

$$(p_1)_* \pi_1 \left(\tilde{X}_1, \tilde{x}_1\right) = (p_2)_* \pi_1 \left(\tilde{X}_2, \tilde{x}_2\right).$$

Proof. If isomorphic, then $p_1 \phi = p_2$ and $p_2 \phi^{-1} = p_1$, so the conclusion follows. Conversely, if these are equal, we can lift the covering maps, and by unique lifting (when basepoints are specified), we get an isomorphism. □

Now suppose that $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via a map $\phi : \tilde{X}_1 \to \tilde{X}_2$. We need to show that the two correspond to the same conjugacy class in $\pi_1(X)$. By the proposition, we have that $(p_1)_* \pi_1 \left(\tilde{X}_1, \tilde{x}_1\right) = (p_2)_* \pi_1 \left(\tilde{X}_2, \phi(\tilde{x}_1)\right)$. However, for any other point $\tilde{x}_2 \in p_2^{-1}(x_0)$, we have that $\pi_1 \left(\tilde{X}_2, \phi(\tilde{x}_1)\right) = \beta_h \left[\pi_1 \left(\tilde{X}_2, \tilde{x}_2\right)\right]$ for some path h since $\tilde{X}_2$ is path connected. Since h is a path from $\phi(\tilde{x}_1)$ to $\tilde{x}_2$, it pushes forward to a loop in X, and so there is an element $g \in \pi_1 (X, x_0)$ such that

$$g^{-1} \left[(p_2)_* \pi_1 \left(\tilde{X}_2, \tilde{x}_2\right)\right] g \subseteq (p_2)_* \pi_1 \left(\tilde{X}_2, \phi(\tilde{x}_1)\right).$$

The other inclusion follows similarly.

Conversely, given a group $H \leq \pi_1 (X, x_0)$ and a conjugate subgroup $g^{-1} H g$, we can lift g and get an isomorphism of the covering space for H. This completes the proof of the theorem.

4. Deck transformations

Definition 17. *An (self-) isomorphism of covering spaces $\tilde{X} \to \tilde{X}$ is called a* deck transformation. *These form a group $G\left(\tilde{X}\right)$.*

Example 1. *For $\mathbb{R} \to S^1$, the deck transformations are the translations of $\mathbb{R}$.*

Example 2. *For an n-sheeted covering $S^1 \to S^1$, the deck transformations form the group $\mathbb{Z}_n$.*

Notice that by the uniqueness of liftings, a deck transformation is uniquely determined by where it sends the basepoint. That says that it is determined by loops in the base! But what is the relationship between loops in the base (subgroups of the fundamental group) and deck transformations?

Definition 18. *A covering space $\tilde{X} \to X$ is normal if for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.*

Example 3. *Look at the coverings of $S^1 \vee S^1$. Which ones are normal?*

Proposition 19. *Let $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let*

$$H = p_* \pi_1 \left(\tilde{X}, \tilde{x}_0\right) \leq \pi_1 (X, x_0).$$

Then:

1. *The group of deck transformations $G\left(\tilde{X}\right)$ is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer subgroup.*
2. *The covering space is normal iff H is a normal subgroup of $\pi_1 (X, x_0)$.*

Corollary 20. *If $\tilde{X}$ is a normal covering, then $G\left(\tilde{X}\right) \cong \pi_1 (X, x_0)/H$. Thus if $\tilde{X}$ is the universal cover, then $G\left(\tilde{X}\right) \cong \pi_1 (X, x_0)$.*

Proof. We saw earlier that changing the basepoint $\tilde{x}_0$ to $\tilde{x}_1$ corresponds to conjugating the $p_* \pi_1 \left(\tilde{X}, \tilde{x}_0\right)$ in $\pi_1 (X, x_0)$ by an element $[\gamma] \in \pi_1 (X, x_0)$. Thus, $[\gamma] \in N(H)$ if

$$p_* \pi_1 \left(\tilde{X}, \tilde{x}_0\right) = p_* \pi_1 \left(\tilde{X}, \tilde{x}_1\right).$$

By the lifting criterion, this is equivalent to the existence of a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. The covering space is normal if there is a complete set of deck transformations, which is equivalent to $N(H) = \pi_1 (X, x_0)$.

We have shown that there is a surjective map $\Phi : N(H) \to G\left(\tilde{X}\right)$ sending $[\gamma]$ to a deck transformation $\tau([\gamma])$ taking $\tilde{x}_0$ to $\tilde{x}_1$ (where $\tilde{\gamma}$ is a lift of γ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tilde{x}_1$). Note that this is a homomorphism: since if $[\gamma], [\gamma'] \in N(H)$ we see that such that $\gamma \cdot \gamma'$ lifts to $\tilde{\gamma} \cdot \tau(\tilde{\gamma}')$. Thus the deck transformation is determined by $\tilde{x}_0$ goes to $\tau(\tau'(\tilde{x}_0))$, and so $[\gamma \cdot \gamma']$ corresponds to $\tau\tau'$.

The kernel of Φ consists of those loops $[\gamma] \in N(H)$ that lift to loops, i.e., precisely $p_* \pi_1 \left(\tilde{X}, \tilde{x}_0\right) = H$. □

Since the universal cover has deck transformation group equal to the fundamental group, if you know the fundamental group, you can construct the universal cover by starting with a neighborhood of the basepoint and then using the fundamental group as group of deck transformations to see the rest of the covering space. Consider $\mathbb{R}^2 \to S^1 \times S^1$.

Example 4. *Consider the universal covering of the Klein bottle, which can be constructed from the fundamental domain by putting two together to get a torus, and then translating to all of $\mathbb{R}^2$. It is not hard to see that the deck transformation group is $\mathbb{Z} \ltimes \mathbb{Z}$, where $(m_1, n_1)(m_2, n_2) = \left(m_1 + (-1)^{n_1} m_2, n_1 + n_2\right)$. Since $\mathbb{R}^2$ is simply connected, this group should be isomorphic to the fundamental group, which we calculated to be $\left\langle a, b \mid abab^{-1} = 1\right\rangle$. We can show that these groups are isomorphic directly to confirm this:*

$$\Phi : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \ltimes \mathbb{Z}$$

is generated by $\Phi(a) = (1,0)$ *and* $\Phi(b) = (0,1)$. *It follows that*

$$\begin{aligned}\Phi(a^m) &= (m,0)\\ \Phi(b^n) &= (0,n)\\ \Phi(a^m b^n) &= (m,n)\\ \Phi\left(abab^{-1}\right) &= (0,0).\end{aligned}$$

Thus we know that Φ *is a surjection and that the normal subgroup generated by* $abab^{-1}$ *is in the kernel. We can see that this is the entire kernel by considering the induced map*

$$\bar{\Phi} : \left\langle a, b \mid abab^{-1} = 1\right\rangle \to \mathbb{Z} \ltimes \mathbb{Z}.$$

Since the relation $ab = ba^{-1}$ *allows one to write any element of the group* $\left\langle a, b \mid abab^{-1} = 1\right\rangle$ *as* $a^m b^n$ *and the map from these elements to* $\mathbb{Z} \ltimes \mathbb{Z}$ *is injective,* $\bar{\Phi}$ *is an isomorphism.*

Note that group actions can be used to construct coverings, but you have to be careful that the regularly covered neighborhoods don't get messed up. The following condition is key:

Definition 21. *A* group action *of* G *on a space* Y *is an injective homomorphism* $G \to Homeo(X)$. *The group action is a* covering space action *if for each* $y \in U$ *there is a neighborhood* U *of* y *such that all the images of* U *are disjoint (i.e.,* $g_1(U) \cap g_2(U) \neq \varnothing$ *implies that* $g_1 = g_1$*).*

The quotient of a space by a covering space action gives a covering space:

Definition 22. *Given a group action* G *on a space* Y, *the* orbit space Y/G *is the space of* orbits $\{Gy : y \in Y\}$ *given the topology of the quotient* $Y/\sim$ *where* $y \sim y'$ *if* $Gy = Gy'$.

Example 5. *For a normal covering space* $\tilde{X} \to X$ *with deck transformation group* $G\left(\tilde{X}\right)$, *we have* $\tilde{X}/G\left(\tilde{X}\right) \approx X$.

Proposition 23. *If* G *is a covering space action on a space* Y, *then:*

(1) *The quotient map* $p : Y \to Y/G$ *is a normal covering space.*
(2) G *is the group of deck transformations if* Y *is path connected.*
(3) G *is isomorphic to* $\pi_1(Y/G)/p_*\pi_1(Y)$ *of* Y *is path connected and locally path connected.*

Proof. Given an open set U as in the definition of covering space action, the quotient will identify the disjoint sets $g(U)$. By the quotient topology, p restricts to a homeomorphism from $g(U)$ to its image $p(U)$ for each $g \in G$, and so p is a covering space. Each element of G acts as a deck transformation, and clearly one can get from any element in the fiber to any other (to get from $g(x_0)$ to $g'(x_0)$, simply use the element $g'g^{-1}$).

Certainly G is a subgroup of the deck transformation group. If Y is path connected, then deck transformations are uniquely determined by where they send a point (by unique lifting!) so G must be the entire deck transformation group.

The last statement follows from our theorem on deck transformations on normal covering spaces. □

Remark 5. *Being a covering space action is related to being a free action and a properly discontinuous action. Free, properly discontinuous actions on a Hausdorff space are covering space actions.*

Corollary 24. *If* Y *is simply connected and* G *is a covering space action on* Y, *then* $\pi_1(Y/G) \cong G$. *Under this isomorphism, a loop in* Y/G *corresponds to an element* $g \in G$ *as the projection of a path from the basepoint* $y_0 \in Y$ *to* $g(y_0)$.

Example 6. *The space* $\mathbb{RP}^n$ *is gotten as* $S^n/\mathbb{Z}_2$, *where the group* $\mathbb{Z}_2$ *is generated by the antipodal map* $x \to -x$. *Note that this is a covering space action. For* $n \geq 2$, S^n *is simply connected, so it follows that* $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ *for* $n \geq 2$. *It is generated by a loop between two antipodal points.*