FUNDAMENTAL GROUP

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1. Introduction

Algebraic topology is mostly about finding invariants for topological spaces. We will primarily be interested in topological invariants that are invariant under certain kinds of smooth deformations called homotopy. In general, we will be able to associate an algebraic object (group, ring, module, etc.) to a topological space. The fundamental group is the simplest, in some ways, and the most difficult in others.

2. Paths and homotopy

Let $I = [0,1]$.

Definition 1. *A* path *in a space X is a continuous map*

$$f : I \to X.$$

We say that f is a path between x_0 and x_1 if $f(0) = x_0$ and $f(1) = x_1$. A homotopy *of paths in X is a continuous map*

$$F : I \times I \to X$$

such that $f_t(s) = F(s,t)$ is a path between x_0 and x_1 for each $t \in I$. Two paths γ_0 and γ_1 are said to be homotopic *if there exists a homotopy F such that*

$$f_0 = \gamma_0, \quad f_1 = \gamma_1.$$

This is often denoted $\gamma_0 \simeq \gamma_1$.

Remark 1. *Sometimes this is called "fixed-endpoint homotopy".*

Example 1. *Linear homotopies in $\mathbb{R}^n$ are given between γ_0 and γ_1 by*

$$f_t(s) = F(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s).$$

In particular, any convex subset of $\mathbb{R}^n$ is homotopic to the constant map.

Proposition 2. *The relation of being homotopic is an equivalence relation on paths with fixed endpoints.*

Proof. Certainly a path is equivalent to itself, since one can take the constant homotopy. It is symmetric as well, since if F is a homotopy from f_0 to f_1, then

$$G(s,t) = F(s, 1-t)$$

Date: January 26, 2011.

is a homotopy from f_1 to f_0 (another way to denote this is f_{1-t}). Suppose $f_0 \simeq f_1$ and $f_1 \simeq f_2$. Then we have homotopies F and G, and then we can take as a homotopy between f_0 and f_2 the scaled map

$$H(s,t) = \begin{cases} F(s,2t) & \text{if } t \leq 1/2 \\ G(s,2t-1) & \text{if } t > \frac{1}{2} \end{cases}$$

This is continuous because $G(s,0) = F(s,1)$. □

Paths can be composed:

Definition 3. *Given paths f and g such that $f(1) = g(0)$, the* composition *or* product path *$f \cdot g$ is given by*

$$f \cdot g(s) = \begin{cases} f(2s) & \text{if } s \leq \frac{1}{2} \\ g(2s-1) & \text{if } x > \frac{1}{2} \end{cases}$$

Because we need the endpoint of one path the be the beginning point for the other for composition, the set of paths does not form a group. However, if we assume that the paths start and end at the same point (loops), then they do. The fundamental group is the set of loops modulo homotopy.

Definition 4. *The fundamental group, $\pi_1(X,x_0)$, is the set of equivalence classes of loops $f : I \to X$ such that $f(0) = f(1) = x_0$.*

Proposition 5. *The fundamental group is a group under composition of loops, i.e., $[f][g] = [f \cdot g]$.*

Proof. Certainly composition is an operation taking loops to loops. We first look to see that composition is well defined on homotopy classes. If $f \simeq f'$ and $g \simeq g'$, then by composing the homotopies we get a homotopy of $f \cdot g$ to $f' \cdot g'$. We can find a homotopy showing that $f \cdot (g \cdot h) \simeq (f \cdot g) \cdot h$. The identity is the equivalence class of the constant loop, $e = [x_0]$. We see that $e \cdot f \simeq f$ and $f \cdot e \simeq f$ by reparametrizing the homotopy. Each loop f has an inverse $f^{-1}(s) = f(1-s)$, which can be seen to be homotopic to e. □

Remark 2. *The constant loop is also an identity in the pseudogroup of paths, i.e., if f is a path and and $f(0) = x_0$, and $f(1) = x_1$. Then $f \cdot x_1 \simeq f$ and $x_0 \cdot f \simeq f$.*

How important is the choice of basepoint?

Proposition 6. *Suppose h is a path from x_0 to x_1 in X. Then h induces an isomorphism between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ by $\beta_h([f]) = \left[h \cdot f \cdot \bar{h}\right]$, where $\bar{h}(s) = h(1-s)$.*

Proof. Certainly β_h is well-defined on homotopy classes. Note that it is a homomorphism since

$$\beta_h([f][g]) = \beta_h([f \cdot g]) = \left[h \cdot f \cdot g \cdot \bar{h}\right] = \left[h \cdot f \cdot h \cdot \bar{h} \cdot g \cdot \bar{h}\right] = \beta_h([f])\,\beta_h([g]).$$

Since β_h has inverse $\beta_{\bar{h}}$, the proposition is proven. □

Thus we often have a choice of basepoint that is essentially irrelevant, and we will refer to the fundamental group as $\pi_1(X)$.

We have the following definition.

Definition 7. *A space X is simply connected if it is path connected and has trivial fundamental group for any choice of basepoint.*

Proposition 8. *A path connected space is simply connected if and only if there is only one homotopy class of paths between any two points.*

Proof. Suppose X is simply connected. Given two paths f, g between x_0 and x_1, we have that

$$f \cdot \bar{g} \simeq e$$
$$\bar{g} \cdot g \simeq e$$

and so

$$f \simeq f \cdot \bar{g} \cdot g \simeq g.$$

Conversely, suppose there is only one homotopy class of paths between any two points. Consider loops based at x_0. Then there is only one homotopy class, and so the fundamental group is trivial. □

3. The circle

In this section we will compute the fundamental group of the circle and some consequences. It is not trivial that the circle has nontrivial fundamental group.

Definition 9. *A space X is contractible if there is a homotopy between the identity map $X \to X$ and a constant map.*

Example 2. $\mathbb{R}^n$ *and the disk* D^n *are contractible. Just consider the homotopy* $F(x,t) = tx$.

It turns out that the circle S^1 is not contractible.

Proposition 10. $\pi_1\left(S^1, (1,0)\right) \cong \mathbb{Z}$.

To prove this, we will need two lifting properties. Let $E : \mathbb{R} \to S^1$ be the map $E(t) = (\cos 2\pi t, \sin 2\pi t)$.

Lemma 11 (Path lifting)**.** *For any path $\gamma : I \to S^1$ and any point $\tilde{x} \in E^{-1}[\gamma(0)] \subseteq \mathbb{R}$, there is a unique path $\tilde{\gamma} : I \to \mathbb{R}$ such that $\tilde{\gamma}(0) = \tilde{x}$ and $\gamma = E \circ \tilde{\gamma}$.*

Lemma 12 (Homotopy lifting)**.** *For any homotopy $F : I \times I \to S^1$ and $\tilde{x} \in E^{-1}(F(0,0))$ $(= E^{-1}(F(0,t))$ for each $t \in I)$, there exists a unique homotopy $\tilde{F} : I \times I \to \mathbb{R}$ such that $\tilde{F}(0,0) = \tilde{x}$ and $F = E \circ \tilde{F}$.*

Let's prove the proposition:

Proof of Proposition 10. Let ω_n be that path $s \to (\cos 2\pi ns, \sin 2\pi ns)$. We will consider the map $\Phi(n) = [\omega_n]$. We first need to see that

$$\omega_{n+m} \simeq \omega_n \cdot \omega_m.$$

To show this, we will use the path lifting property. In particular, there are lifts $\tilde{\omega}_n$, $\tilde{\omega}_m$, and $\tilde{\omega}_{m+n}$ of ω_n, ω_m, and ω_{n+m} such that $\tilde{\omega}_n(0) = \tilde{\omega}_{n+m}(0) = 0$ and $\tilde{\omega}_m(0) = \tilde{\omega}_n(1)$. Then we see that there is a homotopy $\tilde{F}$ between $\tilde{\omega}_{n+m}$ and $\tilde{\omega}_n \cdot \tilde{\omega}_m$ given by

$$\tilde{F}(s,t) = (1-t)\,\tilde{\omega}_{n+m}(s) + t\,(\tilde{\omega}_n \cdot \tilde{\omega}_m)(s).$$

We then see that $E \circ \tilde{F}$ is a homotopy between ω_{n+m} and $\omega_n \cdot \omega_m$. Thus, the map Φ is a homomorphism, since

$$\Phi(n+m) = [\omega_{n+m}] = [\omega_n]\,[\omega_m].$$

Now, to prove that Φ is surjective, we see that given any loop $\gamma : I \to M$, it can be lifted to a path $\tilde{\gamma} : I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$. Since $\tilde{\gamma}$ is a loop, we must have that $\tilde{\gamma}(1) \in \mathbb{Z}$. If we let $n = \tilde{\gamma}(1)$, it is not hard to see that $\tilde{\gamma} \simeq \tilde{\omega}_n$ and hence $\gamma \simeq \omega_n$.

To prove that Φ is injective, let $n, m \in \mathbb{Z}$ such that $\Phi(n) = \Phi(m)$, i.e, $[\omega_n] = [\omega_m]$. Thus there is a homotopy F between ω_n and ω_m, which can be lifted to a homotopy $\tilde{F} : I \times I \to \mathbb{R}$ such that $\tilde{F}(0, \cdot) = 0$ by the homotopy lifting property. Since the endpoint of a homotopy is always the same, we have that $\tilde{F}(1, t)$ is always the same, so that means that $n = m$. $\square$

We can prove both of the lifting properties with a more general lemma.

Lemma 13. *Given a map $F : Y \times I \to S^1$ and a map $\bar{F} : Y \times \{0\} \to \mathbb{R}$ such that $E \circ \bar{F} = F|_{Y \times \{0\}}$, then there is a unique map $\tilde{F} : Y \times I \to \mathbb{R}$ such that $E \circ \tilde{F} = F$ and $\tilde{F}$ extends $\bar{F}$.*

Remark 3. *The property that $E \circ \tilde{F} = F$ can be expressed by saying that $\tilde{F}$ is a* lifting *or* lift *of F.*

We need a certain property of the map E. Specifically:

Claim 1. *There is an open cover $\{U_\alpha\}$ of S^1 such that $E^{-1}(U_\alpha)$ is homeomorphic to a disjoint union of open sets each of which is homeomorphic to U_α.*

Proof. Just take two open arcs that cover S^1. $\square$

Proof of Lemma 13. First, we consider path lifting, where Y is just a point, and we consider F as a map only on I. Note that we can partition I by $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $F([t_i, t_{i+1}])$ is contained in a set U_i such that $E^{-1}(U_i)$ is homeomorphic to a disjoint untion of open sets each of which is homeomorphic to U_i.(The proof is as follows: each point $t \in I$ is contained in an open interval contained in a U_α due to the claim. If we take smaller closed intervals, their interiors form an open cover, and since I is compact, there is a finite subcover, and the closed intervals satisfy this property.) We can now inductively define $\tilde{F}$ since if $\tilde{F}$ is defined up to t_j, then $\tilde{F}(t_j)$ is in some component of $E^{-1}(U_j)$, and that component is homeomorphic to U_j by some homeomorphism $\phi : U_i \to \mathbb{R}$. We then define

$$\tilde{F}\Big|_{[t_i, t_{i+1}]} = \phi \circ F|_{[t_i, t_{i+1}]}.$$

Note that this is unique, since any other map would have to satisfy the same property.

Now suppose Y is not a point. Then we can uniquely lift each point by the previous argument, but that does not ensure continuity. Instead, fpr every point $y \in Y$ there is a neighborhood N such that we can lift the map to $\tilde{F} : N \times I \to \mathbb{R}$. Since the lift is unique at each point, $\tilde{F}$ is actually well-defined on all of Y, since these lifts agree on overlaps. It also follows that $\tilde{F}$ is continuous. $\square$

Lemma 11 follows immediately.

Proof of Lemma 12. First we lift the path $\gamma(s) = F(s, 0)$. Then we use this as the initial condition to lift F to a map $\tilde{F}$. Since $\tilde{F}(\{0\} \times [0, 1])$ is connected and E of it is a single point, $\tilde{F}(0, t)$ must always be the same point. The same is true for $\tilde{F}(1, t)$. $\square$

4. Implications of fundamental group of the circle

Theorem 14 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. We may assume that the polynomial p is of the form

$$\begin{aligned} p(z) &= z^n + a_1 z^{n-1} + \cdots + a_n \\ &= z^n + q(z). \end{aligned}$$

Suppose that $p(z)$ has no roots. Note that this implies that $a_n \neq 0$.

The basic idea is this: we have a homotopy

$$\begin{aligned} f_t(s) &= \frac{p\left(te^{2\pi is}\right)/p(t)}{\left|p\left(te^{2\pi is}\right)/p(t)\right|} \\ f_0(s) &= 1 \\ f_1(s) &= \frac{p\left(e^{2\pi is}\right)/p(1)}{\left|p\left(e^{2\pi is}\right)/p(1)\right|} \end{aligned}$$

between f_0, whose image is $\{1\} \in \mathbb{C}$, and f_1, whose image covers all of S^1. Since $[f_0]$ is the identity in $\pi_1\left(S^1, 1\right)$, it follows that $[f_t]$ is also the identity in $\pi_1\left(S^1, 1\right)$ for any $t \in \mathbb{R}$. We also consider the homotopy

$$\begin{aligned} h_t(s) &= \frac{\left[\left(e^{2\pi is}\right)^n + tq\left(e^{2\pi is}\right)\right]/\left[1 + tq(1)\right]}{\left|\left[\left(e^{2\pi is}\right)^n + tq\left(e^{2\pi is}\right)\right]/\left[1 + tq(1)\right]\right|} \\ h_0(s) &= e^{2\pi isn} = \omega_n(s) \\ h_1(s) &= \frac{p\left(e^{2\pi is}\right)/p(1)}{\left|p\left(e^{2\pi is}\right)/p(1)\right|}. \end{aligned}$$

It follows that ω_n is homotopic to the constant map, which means that $n = 0$, and so p is a constant map.

The problem with this argument is that $h_t(s)$ may not be continuous, since the denominator could be zero. To avoid this, we will choose a very large loop. Instead, consider:

$$\begin{aligned} f_t(s) &= \frac{p\left(t\ re^{2\pi is}\right)/p(t\ r)}{\left|p\left(t\ re^{2\pi is}\right)/p(t\ r)\right|} \\ f_0(s) &= 1 \\ f_1(s) &= \frac{p\left(re^{2\pi is}\right)/p(r)}{\left|p\left(re^{2\pi is}\right)/p(r)\right|}. \end{aligned}$$

Now take r to be large, say $r > \max\left(1, \sum_{j=1}^n |a_j|\right)$. Then

$$\begin{aligned} \left|\left(re^{2\pi is}\right)^n + tq\left(re^{2\pi is}\right)\right| &\geq r^n - t\left|q\left(re^{2\pi is}\right)\right| \\ &> r^n - t\ r^{n-1}\sum |a_j| \\ &\geq (1-t)\, r^n. \end{aligned}$$

In particular,

$$h_t(s) = \frac{\left[\left(re^{2\pi is}\right)^n + tq\left(re^{2\pi is}\right)\right]/\left[r^n + tq(r)\right]}{\left|\left[\left(re^{2\pi is}\right)^n + tq\left(re^{2\pi is}\right)\right]/\left[r^n + tq(r)\right]\right|}$$

is a homotopy between f_1 and ω_n. Thus f_0 is homotopic to ω_n, meaning that $n = 0$. ☐

Definition 15. *Let X be a topological space and $A \subseteq X$ a subspace. Then a map $r : X \to A$ is a* retraction *if $r|_A = \iota_A$, where ι_A is the identity on A. A homotopy $F : X \times I \to A$ is a* deformation retraction *if $F|_{A \times I}$ is the identity, $F|_{X \times \{0\}}$ is the identity, and $F(X \times \{1\}) = A$.*

Let D^n denote the closed disk of radius 1.

Theorem 16. *There is no retraction $D^2 \to S^1$.*

Proof. Suppose $r : D^2 \to S^1$ were a retraction. Let f_0 be a loop in S^1. Since D^2 is contractible, we can define a homotopy in D^2 given by

$$f_t(s) = (1-t) f_0(s) + t x_0,$$

where $x_0 \in S^1$ is the basepoint of f_0. Note that

$$|f_t(s)| \leq (1-t)|f_0(s)| + t|x_0| = 1,$$

so the homotopy stays in D^2. Now consider the homotopy in S^1 given by

$$g_t(s) = r(f_t(s)).$$

Notice that

$$\begin{aligned} g_0(s) &= r(f_0(s)) = f_0(s) \\ g_1(s) &= r(x_0) = x_0. \end{aligned}$$

Thus if there were a retraction, then any loop in S^1 would be contractible to a point. Since this is not true, there is no retraction. ☐

Corollary 17 (2D Brouwer Fixed Point Theorem)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point.*

Proof. Suppose that h did not have a fixed point. Then we can construct a map $r : D^2 \to S^1$ by taking $x \in D^2$ to the point on the ray from $h(x)$ to x that intersects S^1 (since $h(x) \neq x$, there is alway such a ray and it always intersects S^1). This mapping is continuous since small perturbations of x create small perturbations of $h(x)$ and hence small perturbations of $r(x)$. Furthermore, r is a retraction, contradicting the theorem. ☐

Theorem 18 (Borsuk-Ulam Theorem)**.** *For every continuous map $h : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $h(x) = h(-x)$.*

Corollary 19. *S^2 is not homeomorphic to a subset of $\mathbb{R}^2$.*

Proof. By the Borsuk-Ulam Theorem there is no injective map $S^2 \to \mathbb{R}^2$. ☐

5. Products

Proposition 20. *$\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let p_1, p_2 be the projections on the the first and second components. Recall that a map $f : Z \to X \times Y$ is continuous if and only if $p_1 \circ f$ and $p_2 \circ f$ are continuous. Hence a loop γ in $X \times Y$ based at (x_0, y_0) has two loops associated to it, $p_1 \circ \gamma$ and $p_2 \circ \gamma$, which are continuous and properly based. Similarly, a homotopy of the loop γ translates to homotopies of the loops in X and Y. Thus there is a map $\pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$. However, this map is clearly a bijection and a group homomorphism. □

Corollary 21. *The torus $S^1 \times S^1$ has fundamental group $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.*

6. Induced homomorphisms

The previous proof takes maps p_1 and p_2 between spaces and turns them into maps between fundamental groups. This can be done in general.

Definition 22. *A (pointed) map $\phi : (X, x_0) \to (Y, y_0)$ is a map $\phi : X \to Y$ such that $\phi(x_0) = y_0$.*

Proposition 23. *Any map $\phi : (X, x_0) \to (Y, y_0)$ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Proof. The map is simply $\phi_*[\gamma] = [\phi \circ \gamma]$. It is easy to see that this is well-defined and that $\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g)$, so it is a homomorphism. □

Proposition 24. *We have the following two properties of induced homomorphisms:*

(1) $(\phi \circ \psi)_* = \phi_* \psi_*$
(2) $Id_* = Id$

Proof. The first follows from associativity of composition, since

$$(\phi \circ \psi)_* [\gamma] = [\phi \circ \psi \circ \gamma] = \phi_* [\psi \circ \gamma] = \phi_* \psi_* [\gamma].$$

The second is obvious. □

It follows, by the way, that $(\phi^{-1})_* = (\phi_*)^{-1}$.

Proposition 25. $\pi_1(S^n) = \{1\}$ *if $n \geq 2$.*

Proof. Since $S^n \setminus \{pt\}$ is homeomorphic to $\mathbb{R}^n$, which is contractible, if any loop in S^n is homotopic to a loop in $S^n \setminus \{pt\}$, then it is contractible.

Consider a loop $\gamma : I \to S^n$ based at x_0. We choose a point $x_1 \neq x_0$ and let B be a ball around x_1 that does not contain x_0 in its closure. The inverse image $\gamma^{-1}(B)$ is an open set inside $(0, 1)$, and so it is a (possibly infinite) union of intervals. Since $\gamma^{-1}(x_1)$ is compact (why?), it is covered by a finite number of the previously found intervals. Since $[0, 1]$ is compact, for each interval $I_j = (a_j, b_j)$, there is another interval $I'_j = (a'_j, b'_j)$ is defined by

$$a'_j = \inf \{t \leq a_j : \gamma(t) \in B\}$$
$$b'_j = \sup \{t \geq b_j : \gamma(t) \in B\}.$$

This has the property that $\gamma(I'_j) \subseteq B$ and $\gamma(a'_j), \gamma(b'_j) \in \partial B$. Now we simply replace the image of I'_j with paths that stay on the boundary (which can be done if $n \geq 2$, since then the boundary of B is homeomorphic to D^{n-1}, which is path connected. Since the ball B is simply connected, the new path is homotopic to the old, and thus we get a homotopy to a loop in $S^n \setminus \{x_1\}$. □

Proposition 26. $\mathbb{R}^2$ *is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.*

Proof. The case $n = 1$ is easy, since for any map $f : \mathbb{R} \to \mathbb{R}^2$, $\mathbb{R}\backslash \{0\}$ is disconnected, whereas $\mathbb{R}^2 \backslash \{f(0)\}$ is connected. For $n > 2$, we have that $\mathbb{R}^n \backslash \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, so the fundamental groups satisfy

$$\pi_1 (\mathbb{R}^n \setminus \{0\}) \cong \pi_1 \left(S^{n-1}\right) \times \pi_1 (\mathbb{R}) .$$

This group is isomorphic to $\mathbb{Z}$ if $n = 2$ and $\{1\}$ if $n > 2$, so the spaces cannot be homeomorphic. □

Proposition 27. *If a space X retracts onto a subspace A, then the homomorphism*

$$\iota_* : \pi_1 (A, x_0) \to \pi_1 (X, x_0)$$

induced by the inclusion $\iota : A \to X$ is injective. If A is a deformation retract of X, then the map is an isomorphism.

Proof. If $r : X \to A$ is a retraction, then since $r \circ \iota = Id_A$, we have $r_* \iota_* = Id_{\pi_1(A)}$, and so ι_* is injective. If $r_t : X \to X$ is a deformation retraction of X to A, then $r_0 = Id_X$, $r_1 (X) \subseteq A$, and $r_t|_A = Id_A$. If $\gamma : I \to X$ is any loop in X based at x_0, $r_t \circ \gamma$ is a homotopy to a loop based at x_0 in A. Thus ι_* is surjective. □

Remark 4. *This gives an alternate proof that there is no retraction $r : D^2 \to S^1$. If there were, then it would induce an injective homomorphism from $\mathbb{Z}$ to the trivial group, which is impossible.*

Definition 28. *A* basepoint preserving homotopy $\phi_t : (X, x_0) \to (Y, y_0)$ *is a homotopy such that $\phi_t (x_0) = y_0$ for each $t \in I$.*

Proposition 29. *If $\phi_t : (X, x_0) \to (Y, y_0)$ is a basepoint preserving homotopy, then* $(\phi_0)_* = (\phi_1)_*$.

Proof. Given $[\gamma] \in \pi_1 (X, x_0)$, we need to show that $\phi_0 \circ \gamma \simeq \phi_1 \circ \gamma$. Since ϕ_t is basepoint preserving, $\psi_t (s) = \phi_t (\gamma (s))$ is that homotopy. □

Definition 30. *We say that (X, x_0) is* homotopy equivalent *to (Y, y_0), denoted $(X, x_0) \simeq (Y, y_0)$, if there are maps $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (X, x_0)$ such that*

$$\psi \circ \phi \simeq Id_X$$
$$\phi \circ \psi \simeq Id_Y$$

where the homotopies are basepoint preserving.

In fact, we do not need the homotopy to be basepoint preserving.

Proposition 31. *If $\phi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\phi_* : \pi_1 (X, x_0) \to \pi_1(Y, \phi (x_0))$ is an isomorphism.*

The main part of the proof is the following lemma.

Lemma 32. *If $\phi_t : X \to Y$ is a homotopy and h is the path $t \to \phi_t (x_0)$, then* $(\phi_0)_* = \beta_h (\phi_1)_*$.

Proof. We need to show that, given a loop f in X based at x_0, $\phi_0 \circ f \simeq h \cdot (\phi_1 \circ f) \cdot \bar{h}$.
Let

$$h_t (s) = h (st)$$

for $s \in [0,1]$, so that h_t is the path gotten by restricting h to $[0,t]$ and then reparametrizing to be a path on $[0,1]$. Note that $h_t(0) = \phi_0(x_0)$ for all $t \in [0,1]$. Then the for any loop f in X based at x_0, the map

$$F_t(s) = \left(h_t \cdot (\phi_t \circ f) \cdot \bar{h}_t\right)(s)$$

is a homotopy since

$$F_t(0) = \left(h_t \cdot (\phi_t \circ f) \cdot \bar{h}_t\right)(0) = h_t(0) = \phi_0(x_0)$$
$$F_t(1) = \left(h_t \cdot (\phi_t \circ f) \cdot \bar{h}_t\right)(s) = \bar{h}_t(1) = h_t(0) = \phi_0(x_0)$$

and

$$F_0(s) = \left(h_0 \cdot (\phi_0 \circ f) \cdot \bar{h}_0\right)(s)$$
$$F_1(s) = \left(h \cdot (\phi_1 \circ f) \cdot \bar{h}_1\right)(s)$$

so $F_0 \simeq \phi_0 \circ f$. The lemma follows. □

Proof of Proposition 31. Since ϕ is a homotopy equivalence, we have that there is a map $\psi : Y \to X$ and such that $\phi \circ \psi \simeq Id_Y$ and $\psi \circ \phi \simeq Id_X$. By the lemma, it follows that $\psi_* \phi_* = \beta_h$ for some h, and we already know that this is an isomorphism $\pi_1(X, x_0) \to \pi_1(X, \psi \circ \phi(x_0))$. It follows that $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is injective. Similarly, we get that $\phi_* \psi_*$ is an isomorphism $\pi_1(Y, \phi(x_0)) \to \pi_1(Y, \phi \circ \psi \circ \phi(x_0))$, so $\psi_* : \pi_1(Y, \phi(x_0)) \to \pi_1(X, \psi \circ \phi(x_0))$ is injective. since ϕ_* and ψ_* are both injective and their composition is an isomorphism, it follows that both are isomorphisms. □