Math 534B Homework 10.

Due 5/3

1) Lee, Problem 15-3.
2) Lee, Problem 15-7.
3) Lee, Problem 16-2.
4) Usually singular cohomology is defined by using duals at the level of chain complexes. The singular cochain complex with coefficients in a group G consists of groups

$$C^k(X;G) = Hom\left(C^k(X), G\right),$$

and the differential operator for the cochain complex is a map

$$d_k : C^k(X;G) \to C^{k+1}(X;G)$$

defined by

$$(d_k\omega)\left(\sigma^{k+1}\right) = \omega\left(\partial_{k+1}\sigma^{k+1}\right)$$

where $\omega \in C^k(X;G)$ and $\sigma^{k+1} \in C^{k+1}(X)$. The cohomology is then defined in essentially the same way as homology:

$$H^k(X;G) = \ker d_k / \operatorname{Im} d_{k-1}$$

(note the indices for the differential operator are different than for homology, since the differential of a cochain complex increases the index instead of decreasing for a chain complex.) If the G is omitted, it is usually assumed to be $\mathbb{Z}$. Show that there is a well-defined homomorphism

$$\Phi : H^k(X;G) \times H_k(X) \to G$$

given by

$$\Phi([\omega], [\sigma]) = \omega(\sigma).$$

Also show that if $G = \mathbb{R}$ then this map induces an isomorphism

$$H^k(X;\mathbb{R}) \cong Hom(H_k(X), \mathbb{R}).$$

This justifies defining $H^k(X;\mathbb{R})$ to be $Hom(H_k(X), \mathbb{R})$. Note that if $G = \mathbb{Z}$, this may not be true. See the discussion on p. 190 of Hatcher under the heading "The Universal Coefficient Theorem."

More practice: Lee Problem 15-1, 15-2, 15-3, 15-5, 16-1.