Maximum principle and pinching

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1 Introduction

This section will roughly follow Tao's lecture 3. We will look at some basic PDE techniques and apply them to the Ricci flow to obtain some important results about preservation and pinching of curvature quantities. The important fact is that the curvatures satisfy certain reaction-diffusion equations which can be studied with the maximum principle.

2 The maximum principle

Recall that if a smooth function $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ has a local minimum at x_0 in the interior of U, then

$$\frac{\partial u}{\partial x^i}(x_0) = 0$$
$$\frac{\partial^2 u}{\partial x^i \partial x^j}(x_0) \geq 0$$

where the second statement is that the Hessian is nonnegative definite (has all nonnegative eigenvalues). The same is true on a Riemannian manifold, replacing regular derivatives with covariant derivatives.

Lemma 1 *Let (M, g) be a Riemannian manifold and $u : M \to \mathbb{R}$ be a smooth (or at least C^2) function that has a local minimum at $x_0 \in M$. Then*

$$\begin{aligned} \nabla_i u(x_0) &= 0 \\ \nabla_i \nabla_j u(x_0) &\geq 0 \\ \triangle u(x_0) &= g^{ij}(x_0) \nabla_i \nabla_j u(x_0) \geq 0. \end{aligned}$$

Proof. In a coordinate patch, the first statement is clear since $\nabla_i u = \frac{\partial u}{\partial x^i}$. The second statement is that the Hessian is positive definite. Recall that in coordinates, the Hessian is

$$\nabla_i \nabla_j u = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k},$$

but at a minimum, the second term is zero and the positive definiteness follows from the case in $\mathbb{R}^n$. The last statement is true since both g and the Hessian are positive definite. ■

Remark 2 *There is a similar statement for maxima.*

The following lemma is true in the generality of a smooth family of metrics, though is also of use for a fixed metric.

Lemma 3 *Let $(M, g(t))$ be a smooth family of compact Riemannian manifolds for $t \in [0, T]$. Let $u : [0, T] \times M \to \mathbb{R}$ be a C^2 function such that*

$$u(0, x) \geq 0$$

for all $x \in M$. Also let $A \in \mathbb{R}$. Then exactly one of the following is true:

1. *$u(t, x) \geq 0$ for all $(t, x) \in [0, T] \times M$, or*
2. *There exists a $(t_0, x_0) \in (0, T]$ such that all of the following are true:*

$$\begin{aligned} u(t_0, x_0) &< 0 \\ \nabla_i u(t_0, x_0) &= 0, \\ \triangle_{g(t_0)} u(t_0, x_0) &\geq 0, \\ \frac{\partial u}{\partial t}(x_0, t_0) &\leq 0. \end{aligned}$$

Proof. Certainly both cannot hold. Now suppose 1 fails. Then there must exist $\varepsilon > 0$ and (t_0, x_0) such that $u(t_0, x_0) < 0$. We may move this to the minimum point, at which all of the first three must hold. If we take this to be the first time that such a point occurs, the last must hold as well. ■

Corollary 4 *Let $(M, g(t))$ be a smooth family of compact Riemannian manifolds for $t \in [0, T]$. Let $u, v : [0, T] \times M \to \mathbb{R}$ be C^2 functions such that*

$$u(0, x) \geq v(0, x)$$

for all $x \in M$. Also let $A \in \mathbb{R}$. Then exactly one of the following is true:

1. *$u(t, x) \geq v(t, x)$ for all $(t, x) \in [0, T] \times M$, or*
2. *There exists a $(t_0, x_0) \in (0, T]$ such that all of the following are true:*

$$\begin{aligned} u(t_0, x_0) &< v(t_0, x_0) \\ \nabla_i u(t_0, x_0) &= \nabla_i v(t_0, x_0), \\ \triangle_{g(t_0)} u(t_0, x_0) &\geq \triangle_{g(t_0)} v(t_0, x_0), \\ \frac{\partial u}{\partial t}(x_0, t_0) &\leq \frac{\partial v}{\partial t}(t_0, x_0) + A[u(t_0, x_0) - v(t_0, x_0)]. \end{aligned}$$

Proof. Replace u with $e^{-At}(u-v)$. ■

This will allow us to estimate subsolutions of a heat equation by supersolutions of the same heat equation.

Corollary 5 *Let the assumptions be the same as in Corollary 4, including*

$$u(0,x) \geq v(0,x).$$

Suppose u is a supersolution of a reaction-diffusion equation, i.e.,

$$\frac{\partial u}{\partial t} \geq \triangle_{g(t)} u + \nabla_{X(t)} u + F(t,u)$$

and v is a subsolution of the same equation, i.e.,

$$\frac{\partial v}{\partial t} \leq \triangle_{g(t)} u + \nabla_{X(t)} v + F(t,v)$$

for all $(t,x) \in [0,T] \times M$, where $X(t)$ is a vector field for each t and $F(t,w)$ is Lipschitz in w, i.e., there is $A > 0$ such that

$$|F(t,w) - F(t,w')| \leq A |w - w'|.$$

Then

$$u(t,x) \geq v(t,x)$$

for all $t \in [0,T]$.

Proof. Consider

$$\begin{aligned}\frac{\partial}{\partial t}(u-v) &\geq \triangle_{g(t)}(u-v) + \nabla_{X(t)}(u-v) + F(t,u) - F(t,v) \\ &\geq \triangle_{g(t)}(u-v) + \nabla_{X(t)}(u-v) - A|u-v|.\end{aligned}$$

The dichotomy in Corollary 4 says that either $u - v \geq 0$ for all t, x or else there is a point (t_0, x_0) such that at that point,

$$\begin{aligned}u - v &< 0 \\ \triangle(u-v) &= 0 \\ \nabla(u-v) &= 0 \\ \frac{\partial}{\partial t}(u-v) &\leq A'(u-v) = -A'|u-v|\end{aligned}$$

for any A'. But the inequality above says that at that same point

$$\frac{\partial}{\partial t}(u-v) \geq -A|u-v|,$$

which is a contradiction if $-A' < -A$. ■

Usually, instead of making v a subsolution, we will just make v the subsolution to the ODE

$$\frac{dv}{dt} \leq F(t,v),$$

where $v = v(t)$ is independent of x and so this is also a subsolution to the PDE.

Here is an easy application:

Proposition 6 *Nonnegative scalar curvature is preserved by the Ricci flow, i.e., if $R(0,x) \geq 0$ for all $x \in M$ and the metric g satisfies the Ricci flow for $t \in [0,T)$, then $R(t,x) \geq 0$ for all $x \in M$ and $t \in [0,T]$.*

Proof. Recall that R satisfies the evolution equation

$$\frac{\partial R}{\partial t} = \triangle_{g(t)} R + 2\left|\mathrm{Rc}\right|^2,$$

thus it is a supersolution to the heat equation (with changing metric), i.e.,

$$\frac{\partial R}{\partial t} \geq \triangle R.$$

By Corollary 5, we must have that $R \geq 0$ for all t. ■

We can actually do better. Notice that if T_{ij} is a 2-tensor on an n-dimensional Riemannian manifold (M,g), then

$$|T|^2 \geq \frac{1}{n}\left(g^{ij}T_{ij}\right)^2$$

since

$$\left|T_{ij} - \frac{1}{n}\left(g^{k\ell}T_{k\ell}\right)g_{ij}\right|^2 \geq 0$$

(expand that out and see it implies the previous inequality). Thus

$$\left|\mathrm{Rc}\right|^2 \geq \frac{1}{n}R^2$$

and so scalar curvature satisfies

$$\frac{\partial R}{\partial t} \geq \triangle R + \frac{2}{n}R^2.$$

The maximum principle implies that $R(t,x) \geq f(t)$ for all $x \in M$, where $f(t)$ is the solution to the ODE

$$\frac{df}{dt} = \frac{2}{n}f^2$$
$$f(0) = \min_{x \in M} R(x,0).$$

This equation can be solved explicitly as

$$\int \frac{1}{f^2}df = \int \frac{2}{n}dt$$
$$-\frac{1}{f} = \frac{2}{n}t - \frac{1}{f(0)}$$
$$f(t) = \frac{f(0)}{1 - \frac{2}{n}f(0)\,t}$$

as long as $f(0) \neq 0$. Notice that if $f(0) > 0$ then this says that $R(t,x)$ goes to infinity in finite time $T \leq \frac{n}{2f(0)}$. If $f(0) < 0$, then this says that if the flow exists for all time, then the scalar curvature becomes nonnegative in the limit.

3 Maximum principle on tensors

Sometimes it may be useful to use a tensor variant, for a function $u : [0, T] \to \Gamma(V)$ where $\Gamma(V)$ are sections of a tensor bundle (such as if we wish to apply the maximum principle to the Ricci tensor, for instance). Here is the theorem (possibly due to Hamilton?)

Lemma 7 *Let (M, g) be a d-dimensional Riemannian manifold and let V be a vector bundle over M with connection ∇. Let K be a closed, fiberwise convex subset of V which is parallel with respect to the connection. Let $u \in \Gamma(V)$ be a section such that*

1. *$u(x) \in \partial K_x$ at some point $x \in M$, and*
2. *$u(y) \in K_y$ for all y in a neighborhood of x*

(This is the notion that u attains a maximum at x.) Then $\nabla_X u(x)$ is tangent to K_x at $u(x)$ and the Laplacian $\triangle u(x) = g^{ij}(x) \nabla_i \nabla_j u(x)$ is an inward or tangential pointing vector to K_x at $u(x)$.

Here are the relevant definitions.

Definition 8 *A subset K of a tensor bundle $\pi : E \to M$ is fiberwise convex if the fiber $K_x = K \cap E_x$ (where $E_x = \pi^{-1}(x)$) is a convex subset of the vector space E_x.*

Definition 9 *A subset K is parallel to the connection ∇ if it is preserved by parallel translation, i.e., if $P_{x,y}$ is parallel translation along a curve from x to y, then $P^*_{x,y} K_y \subset K_x$ (this is if the tensors are all contravariant).*

Example 10 *The set of positive definite two-tensors is fiberwise convex and parallel with respect to the Levi-Civita connection.*

The maximum principle on tensors can be used to show things like:

1. Nonnegative Ricci curvature is preserved by Ricci flow in dimension 3.
2. Nonnegative curvature operator is preserved by Ricci flow in all dimensions.

We will go into this in more detail in future lectures.