Estimates on the Metric Tensor

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1. Introduction. This is a summary of some easy estimates on the metric tensor of a manifold if the sectional curvatures are bounded. It generally follows [**1**, section 1.8].

2. Definitions. We will estimate derivatives of the metric in geodesic polar coordinates. Suppose we have a Riemannian manifold (M, g). We consider geodesic polar coordinates, and in these coordinates the metric can be written as $g = dr^2 + r^2 g_{\theta^i \theta^j} d\theta^i d\theta^j$.

We also are going to need to take a geodesic $\gamma(t)$ on M which is parametrized by arclength, and we will say that $P = \gamma(0)$, $Q = \gamma(r)$, and $T(t) = \gamma'(t)$.

In the sequel, we shall talk about vector fields V along the curve γ. When we write $V'(t)$, it is understood to mean $\nabla_{T(t)} V(t)$, where ∇ is the Riemannian connection on M.

We shall also restrict ourselves to r such that $\gamma(r)$ is in the ball around P where $\exp_P$ is a diffeomorphism (inside the injectivity radius).

3. Relevant Theorems. We will use Jacobi fields and the Index Inequality. We first should define the Index of a vector field.

Definition 3.1. The index at r of a vector field $V(t)$ along a curve γ is

$$I_r(V, V) = \int_0^r \left[g(V'(t), V'(t)) + g(R(T(t), V(t))T(t), V(t)) \right] dt$$

where $T(t) = \frac{d\gamma}{dt}$ is the tangential vector field to γ.

This is, I think, essentially the second variation of arclength.

The important theorem is the following:

Theorem 3.2 (Index Inequality. See, for instance, [**2**]). *Let $\gamma(t)$ be a geodesic between $P = \gamma(0)$ and $Q = \gamma(r)$ such that there are no conjugate points between P and Q, and let $J(t)$ be a Jacobi field along γ such that $J(0) = 0$ and $V(t)$ be another smooth vector field along γ such that both J and V are orthogonal to $T(t) = \frac{d\gamma}{dt}$, $V(0) = 0$, and $J(r) = V(r)$, then $I_r(J, J) \leq I_r(V, V)$.*

In order to use the Index Inequality, we will need the following two propositions.

Proposition 3.3 (See [**1**], 1.46). *$Q = \exp_P X_0$ is a conjugate point if and only if $\exp_P$ is singular at X_0.*

Thus if we stay within the injectivity radius on M, we will not encounter any conjugate points. We will also need to know:

Proposition 3.4. *The injectivity radius for a space of constant curvature b^2 is π/b.*

Finally, we will use the fact that Jacobi fields are the infinitesimal variations of geodesics. In geodesic normal coordinates at P, the geodesics through p are given by $exp_p(tv)$. Thus if we have a Jacobi field along a geodesic γ such that $J(0) = 0$, then we see that in our geodesic polar coordinates we have Jacobi fields $J(t) = t\|J'(0)\| \left.\frac{\partial}{\partial\theta}\right|_{tv}$ where $\|w\|$ for a vector w means $\sqrt{g(w,w)}$. Thus we see that:

Proposition 3.5. *For a Jacobi field J along a geodesic $\gamma(t)$, $t \in [0,r]$ with $\gamma(r) = Q$, such that $J(0) = 0$ and $J(r) = \frac{\partial}{\partial\theta}|_Q$, we have $g(J(r), J(r)) = r^2 g_{\theta\theta}\|J'(0)\|^2$.*

4. Lower Bounds for the Components of the Metric Tensor.

We shall prove the following:

Theorem 4.1. *If the sectional curvature of M is bounded above by b^2 then we have the following two estimates, provided $r < \pi/b$:*

- $\frac{\partial}{\partial r}\log\sqrt{g_{\theta\theta}} \geq b\cot(br) - \frac{1}{r}$
- $g_{\theta\theta}(r,\Theta) \geq \frac{\sin br}{br}$

where $\theta = \theta^i$ for any i (as defined in the polar representation of the metric).

Now suppose the sectional curvature of M is bounded above by b^2, so $g(R(J,T)T,J) \leq b^2$, or $g(R(T,J)T,J) \geq -b^2$.

We furthermore observe that if J is a Jacobi field, then, by definition, $J''(t) = R(T,J)T$, so we get

$$I_r(J,J) = \int_0^r [g(J'(t),J'(t)) + g(J''(t),J(t))]\,dt = g(J'(r),J(r)) \tag{4-1}$$

We are going to compare the metric on M with the metric on a space of constant curvature b^2. To do this, we need to make a frame $\{e_1(t),...,e_n(t)\}$ on M by taking an orthonormal basis $\{e_1,...,e_n\}$ of $T_{\gamma(0)}M$ such that $e_1 = \gamma'(0)$ and then parallel translating it along γ. Notice that since γ is a geodesic and parametrized by arclength, $e_1(t) = \gamma'(t)$. We also need to take a geodesic $\tilde{\gamma}$ on another manifold $\tilde{M}$ which is the same length as γ and do the same thing to get a frame $\{\tilde{e}_1(t),...,\tilde{e}_n(t)\}$ along $\tilde{\gamma}$. We then have the map $\sim$: $c^i(t)e_i(t) \mapsto c^i(t)\tilde{e}_i(t)$, or $V \mapsto \tilde{V}$. Notice the following:

Lemma 4.2. *The map $\sim$: $(M,g) \to (\tilde{M},\tilde{g})$ has the following properties for vector fields V and W along γ and $\tilde{V}$ and $\tilde{W}$ along $\tilde{\gamma}$:*

- $g(V,W) = \tilde{g}(\tilde{V},\tilde{W})$
- $g(V',W) = \tilde{g}(\tilde{V}',\tilde{W})$
- $g(V',W') = \tilde{g}(\tilde{V}',\tilde{W}')$

Proof: Use the frames we have already set up. Then if $V = v^i e_i$ and $W = w^i e_i$ where v^i, w^i are real-valued functions, then

$$g(V,W) = v^i w^j g(e_i,e_j) = \sum v^i w^i = v^i w^j \tilde{g}(\tilde{e}_i,\tilde{e}_j) = \tilde{g}(\tilde{V},\tilde{W})$$

(Notice that it is important that we use *frames* instead of *coordinates*.) Furthermore, since the frames are parallel, i.e. $\nabla_T e_i = 0$,, we get $V' = v^{i\prime} e_i + v^i \nabla_T e_i = v^{i\prime} e_i$ and $\tilde{V}' = v^{i\prime} \tilde{e}_i$ so the last items follow. □

Now,

$$\begin{aligned} I(J,J) &= \int_0^r [g(J',J') + g(R(T,J)T,J)]\,dt \\ &\geq \int_0^r \left[g(J',J') - b^2 g(J,J)\right]dt \\ &= I^b(\tilde{J},\tilde{J}) \end{aligned}$$

where I^b is the index on a manifold of constant curvature b^2 and where we have chosen a frame $\{e_i\}$ along γ and a corresponding frame $\{\tilde{e}_i\}$ along some curve $\tilde{\gamma}$ of the same length in the manifold of constant curvature.

In constant curvature, we have a Jacobi field which vanishes at $t = 0$ and ends at the vector $\tilde{J}(r)$ given by $\frac{\sin bt}{\sin br}\tilde{J}(r)$.

What do we really mean by this? Suppose $J(r) = \sum c^i e_i(r)$ with our frame as above. Let our new field be defined by $\tilde{V}(t) = \frac{\sin bt}{\sin br} c^i(r)\tilde{e}_i(t)$. Notice that $\tilde{V}(r) = \tilde{J}(r)$, $\tilde{V}(0) = 0$, $\tilde{V}'(t) = b\frac{\cos bt}{\sin br} c^i(r)\tilde{e}_i(t)$ since the frame is parallel, and $\tilde{V}$ is orthogonal to $\tilde{\gamma}$ at every point since $J(r)$ is and we have a parallel frame along a geodesic, so $\tilde{e}_1(t) = \tilde{T}(t) = \tilde{\gamma}'(t)$ for all t. Now compute (letting $c_i = c_i(r)$) using the Index Inequality (3.2) on the manifold of constant curvature b^2 (which we can do since we are inside the injectivity radius by assumption that $r \leq \pi/b$):

$$\begin{aligned} I_r^b(\tilde{J},\tilde{J}) &\geq I_r^b\left(\frac{\sin(bt)}{\sin(br)} c^i \tilde{e}_i(t)\right) \\ &= \frac{1}{\sin^2 br}\int_0^r \left[b^2\cos^2(bt)\sum (c^i)^2 - b^2\sin^2(bt)\sum (c^i)^2\right]dt \\ &= \frac{b}{\sin^2 br}\sin(br)\cos(br)\sum (c^i)^2 \\ &= b\cot(br) g(J(r),J(r)) \end{aligned}$$

We have just about proven an estimate on the metric tensor. We use 4-1. We now know that

$$g(J'(r),J(r)) \geq b\cot(br) g(J(r),J(r)) \tag{4-2}$$

Now if we take $J(t)$ to be a Jacobi field such that $J(r) = \frac{\partial}{\partial\theta}|_Q$ for $\theta = \theta^i$ for some i, then $g(J(r),J(r)) = r^2 g_{\theta\theta}\|J'(0)\|^2$ by Proposition 3.5, so we can compute:

$$\begin{aligned} b\cot(br) g(J(r),J(r)) &\leq g(J'(r),J(r)) \\ b\cot(br) &\leq \frac{g(J'(r),J(r))}{g(J(r),J(r))} \\ &= \frac{\frac{\partial}{\partial r} g(J(r),J(r))}{2g(J(r),J(r))} \end{aligned}$$

$$= \frac{\frac{\partial}{\partial r}(r^2 g_{\theta\theta}\|J'(0)\|^2)}{2r^2 g_{\theta\theta}\|J'(0)\|^2}$$

$$= \frac{1}{r} + \frac{\partial}{\partial r}\log\sqrt{g_{\theta\theta}}$$

And thus we have our first estimate on the metric tensor in polar coordinates:

$$\frac{\partial}{\partial r}\log\sqrt{g_{\theta\theta}} \geq b\cot(br) - \frac{1}{r}$$

Notice that

$$b\cot(br) - \frac{1}{r} = \frac{d}{dr}\log\frac{\sin(br)}{r} = \frac{d}{dr}\left(\log\frac{\sin(br)}{br} + \log b\right)$$

so if we integrate from 0 to r_0 we get

$$\log\sqrt{g_{\theta\theta}(r_0,\Theta)} \geq \log\frac{\sin br_0}{br_0}$$

since at 0, the metric is Euclidean, so $g_{\theta\theta}(0,\Theta) = 1$ (since in $\mathbb{R}^2$ we have $g(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) = r^2$) and $\lim_{s\to 0}\frac{\sin br}{br} = 1$. Thus we have

$$g_{\theta\theta}(r,\Theta) \geq \frac{\sin br}{br}$$

for any θ, a coordinate on S^{n-1}.

5. Lower Bounds on Determinant of Metric. We shall now use what we proved above to prove:

Theorem 5.1. *If the sectional curvature of M is bounded above by b^2 then we have the following two estimates, provided $r < \pi/b$:*

- $\frac{\partial}{\partial r}\log\sqrt{|g(r,\theta)|} \geq (n-1)b\cot(br) - \frac{n-1}{r}$
- $\sqrt{|g(r,\theta)|} \geq \left(\frac{\sin br}{br}\right)^{n-1}$

where $|g| = \det g_{\theta^i\theta^j}$.

We first take a bunch of Jacobi Fields $J_1,\ldots,J_{n-1}$ such that together with T at r they form a basis for T_QM. We now let $|g| = \det g_{\theta^i\theta^j}$ and suppose that in coordinates we have $J_i(t) = c_i^k\frac{\partial}{\partial x^k}$ and compute:

$$\begin{aligned}
\frac{\partial}{\partial r}\log\sqrt{|g|} &= \frac{1}{2}\frac{\partial}{\partial r}\mathrm{tr}\log|g| \\
&= \frac{1}{2}tr\left(\frac{\partial}{\partial r}g_{\theta^i\theta^j}g^{\theta^j\theta^k}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\frac{g(J_i(r),J_j(r))}{r^2}g^{\theta^i\theta^j} \\
&= g(J_i'(r),J_j(r))g^{\theta^i\theta^j} - \frac{n-1}{r} \\
&= \sum_i g(J_i'(r),J_i(r)) - \frac{n-1}{r}
\end{aligned}$$

We get the last equality seen above by letting $J'_i(r) = c_i^k J_k(r)$ (since the $J'_k(r)$ form an orthonormal basis) and compute:

$$
\begin{aligned}
g(J'_i(r), J_j(r))g^{\theta^i\theta^j} &= c_i^k r^2 g_{\theta^k\theta^j} g^{\theta^i\theta^j} \\
&= r^2 c_i^k \delta_k^i \\
&= \sum_i r^2 c_i^i \\
&= \sum_i g(J'_i(r), J_i(r))
\end{aligned}
$$

We can now use 4-2 and see that

$$
\frac{\partial}{\partial r} \log \sqrt{|g|} \geq (n-1) b \cot(br) - \frac{n-1}{r}
$$

Integrating this from 0 to r_0 we get

$$
\begin{aligned}
\int_0^{r_0} \left[(n-1) b \cot(br) - \frac{n-1}{r} \right] dr &= \log \left(\frac{\sin br}{br} \right)^{n-1} \Bigg|_0^{r_0} \\
&= \log \left(\frac{\sin br_0}{br_0} \right)^{n-1}
\end{aligned}
$$

because $\lim_{r \to 0} \frac{\sin br}{br} = 1$. Now, when $r = 0$, g is the Euclidean metric, where $g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = r^2$, so $\|g(0, \Theta)\| = \det I_{n-1} = 1$ where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Thus we finally get:

$$
\begin{aligned}
\log \sqrt{|g(r_0, \Theta)|} &\geq \log \left(\frac{\sin br_0}{br_0} \right)^{n-1} \\
\sqrt{|g(r_0, \Theta)|} &\geq \left(\frac{\sin br_0}{br_0} \right)^{n-1}
\end{aligned}
$$

6. Upper Bounds on Components of the Metric Tensor. We shall prove the following:

Theorem 6.1. *If the sectional curvature of M is bounded below by $-a^2$ then we have the following two estimates:*

- $\frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} \leq a \coth(ar) - \frac{1}{r}$
- $g_{\theta\theta}(r, \Theta) \leq \frac{\sinh ar}{ar}$

We begin as above. We know that for a Jacobi field J along a geodesic we have $g(R(J,T)T, J) \geq -a^2$, so $g(R(T,J)T, J) \leq a^2$.

Let us define a vector field $V(t)$ along the geodesic γ. We first consider a Jacobi field J along γ such that $J(0) = 0$, $g(J'(0), T(0)) = 0$, and $J(r) = \frac{\partial}{\partial \theta}|_Q$. We fix a frame $\{e_i(t)\}$ along γ as above, so $J(t) = c^i(t) e_i(t)$. We can now take as our vector field $V(t) = \frac{\sinh at}{\sinh ar} c^i(r) e_i(t)$.

Notice that $\tilde{V}$ would be a Jacobi field in a manifold $\tilde{M}$ of constant curvature $-a^2$. We also see immediately that $V(0) = 0$ and $V(r) = J(r)$.

The Index Inequality (3.2) gives us

$$\begin{aligned} I_r(J, J) &\leq I_r(V, V) \\ &= \int_0^r [g(V', V') + g(R(T, V)T, V)]\, dt \\ &\leq \int_0^r \left[g(V'(t), V'(t)) + a^2 g(V, V)\right] dt \\ &= I_r^a(\tilde{V}, \tilde{V}) \end{aligned}$$

where I_r^a is the index on a manifold of constant curvature $-a^2$. Note we can use the Index Inequality since we are within the injectivity radius and hence there are no conjugate points.

Now, recall 4-1, which handles the left side. We also want to compute the right side (again letting $c^i = c^i(r)$):

$$\begin{aligned} I_r^a(\tilde{V}, \tilde{V}) &= \int_0^r \left[g(V'(t), V'(t)) + a^2 g(V, V)\right] dt \\ &= \frac{1}{\sinh^2 ar} \int_0^r \left[a^2 \cosh^2(at) \sum (c^i)^2 + a^2 \sinh^2(at) \sum (c^i)^2\right] dt \\ &= \frac{a}{\sinh^2 ar} \int_0^r \left[a \cosh^2 at + a \sinh^2 at\right] dt \sum (c^i)^2 \\ &= \frac{a}{\sinh^2 ar} (\cosh ar)(\sinh ar) g(J(r), J(r)) \\ &= a \coth(ar) g(J(r), J(r)) \end{aligned}$$

Now we can follow the same calculation as above to get the following two estimates:

$$\frac{\partial}{\partial r} \log \sqrt{g_{\theta\theta}} \leq a \coth(ar) - \frac{1}{r}$$

$$g_{\theta\theta}(r, \Theta) \leq \frac{\sinh ar}{ar}$$

References

[**1**] Thierry Aubin. *Some Nonlinear Problems in Riemannian Geometry.* Springer-Verlag, Berlin, Germany, 1998.

[**2**] Manifred Perdigao do Carmo. *Riemannian Geometry.* Birkhauser, Boston, MA, 1992.