Composite Hypotheses

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For a **composite hypothesis**, the parameter space Θ is divided into two disjoint regions, Θ_0 and Θ_1. The test is written

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

with H_0 is called the null hypothesis and H_1 the alternative hypothesis. Consequently, the action space A has two points 0 and 1 and the decision or **test function**

$$d : \text{data} \to \{0, 1\}.$$

Type I and type II errors have the same meaning for a composite hypotheses as it does with a simple hypothesis.

1 Power

Power is now a function

$$\pi_d(\theta) = P_\theta\{d(X) = 1\}.$$

that gives the probability of rejecting the null hypothesis for a given value of the parameter. Consequently, the ideal test function has

$$\pi_d(\theta) = 0 \text{ for all } \theta \in \Theta_0 \text{ and } \pi_d(\theta) = 1 \text{ for all } \theta \in \Theta_1$$

and the test function yields the correct decision with probability 1.

In reality, incorrect decisions are made. For $\theta \in \Theta_0$,

$$\pi_d(\theta) \;\; \text{is the probability of making a type I error}$$

and for $\theta \in \Theta_1$,

$$1 - \pi_d(\theta) \;\; \text{is the probability of making a type II error.}$$

The goal is to make the chance for error small. The traditional method is the same as that employed in the Neyman-Pearson lemma. Fix a **level** α, defined to be

$$\alpha = \sup\{\pi_d(\theta); \theta \in \Theta_0\}$$

and look for a decision function that make the power function large for $\theta \in \Theta_1$

Example 1. *For $X_1, X_2, \ldots, X_n$ independent $U(0,\theta)$ random variables, $\theta \in \Theta \in [0,\infty)$. Take*

$$H_0 : \theta_L \leq \theta \leq \theta_R \quad \text{versus} \quad H_1 : \theta < \theta_L \text{ or } \theta > \theta_R.$$

We will try to base a test based on the sufficient statistic $X_{(n)} = \max_{1\leq i\leq n} X_i$ and reject H_0 if $X_{(n)} > \theta_R$ and too much smaller that θ_L, say $\tilde{\theta}$. Then, the power function

$$\pi_d(\theta) = P_\theta\{X_{(n)} \leq \tilde{\theta}\} + P_\theta\{X_{(n)} \geq \theta_R\}$$

We compute the power function in three cases.

Case 1. $\theta \leq \tilde{\theta}$.

$$P_\theta\{X_{(n)} \leq \tilde{\theta}\} = 1 \text{ and } P_\theta\{X_{(n)} \geq \theta_R\} = 0$$

and therefore $\pi_d(\theta) = 1$.

Case 2. $\tilde{\theta} < \theta \leq \theta_R$.

$$P_\theta\{X_{(n)} \leq \tilde{\theta}\} = \left(\frac{\tilde{\theta}}{\theta}\right)^n \text{ and } P_\theta\{X_{(n)} \geq \theta_R\} = 0$$

and therefore $\pi_d(\theta) = (\tilde{\theta}/\theta)^n$.

Case 3. $\theta > \theta_R$.

$$P_\theta\{X_{(n)} \leq \tilde{\theta}\} = \left(\frac{\tilde{\theta}}{\theta}\right)^n \text{ and } P_\theta\{X_{(n)} \geq \theta_R\} = 1 - \left(\frac{\theta_R}{\theta}\right)^n$$

and therefore $\pi_d(\theta) = (\tilde{\theta}/\theta)^n + 1 - (\theta_R/\theta)^n$.

The size of the test

$$\alpha = \sup\left\{\left(\frac{\tilde{\theta}}{\theta}\right)^n ; \theta_L \leq \theta \leq \theta_R\right\} = \left(\frac{\tilde{\theta}}{\theta_L}\right)^n.$$

To achieve this level, choose $\tilde{\theta} = \theta_L \sqrt[n]{\alpha}$.

Example 2. *Let $X_1, X_2, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables with σ^2 and μ unknown. For the composite hypothesis for the* **one-sided test**

$$H_0 : \mu \leq \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0.$$

We use the test statistic from the likelihood ratio test and reject H_0 if $\bar{X}$ is too large. The power function

$$\pi_d(\mu) = P_\theta\{\bar{X} \geq k(\mu_0)\}.$$

To obtain level α, we want $\alpha = \pi_d(\mu_0)$ then

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = z_\alpha.$$

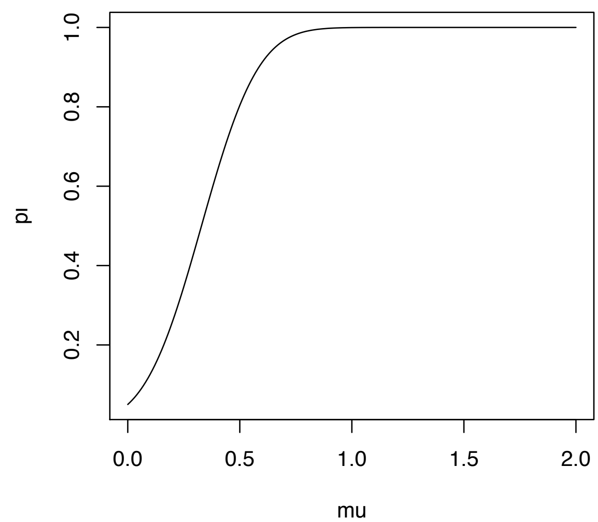

Figure 1: Power function for the one-sided test above

where $\Phi(z_\alpha) = 1-\alpha$ *and* Φ *is the distribution function for the standard normal, thus* $k(\mu_0) = \mu_0 + (\sigma/\sqrt{n})z_\alpha$.

The power function for this test

$$\begin{aligned}\pi_d(\mu) &= P_\mu\{\bar{X} \geq \frac{\sigma}{\sqrt{n}} z_\alpha + \mu_0\} = P_\mu\{\bar{X} - \mu \geq \frac{\sigma}{\sqrt{n}} z_\alpha - (\mu - \mu_0)\} \\ &= P_\mu\left\{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \geq z_\alpha - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right\} = 1 - \Phi\left(z_\alpha - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)\end{aligned}$$

We plot the power function with $\mu_0 = 0$, $\sigma = 1$, *and* $n = 25$,

```
> zalpha=qnorm(.95)
> mu=(0:200)/100
> z=zalpha-5*mu
> pi=1-pnorm(z)
> plot(mu,pi,type="l")
```

For a **two-sided test**

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

We reject H_0 *if* $|\bar{X} - \mu_0|$ *is too large. Again, to obtain level* α,

$$|Z| = \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| = z_{\alpha/2}.$$

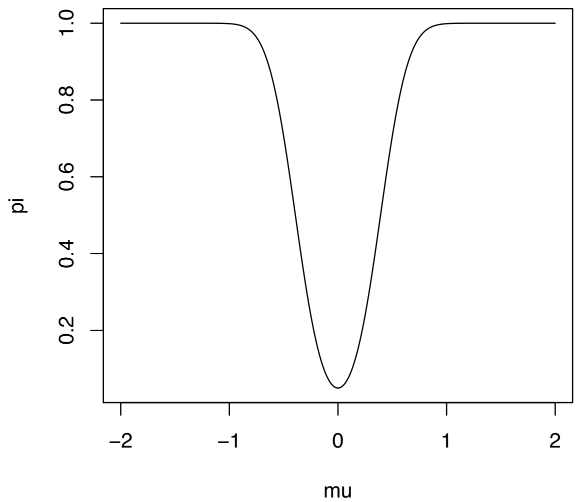

Figure 2: Power function for the two-sided test above

The power function for the test

$$\begin{aligned}\pi_d(\mu) &= 1 - P_\mu\left\{-z_{\alpha/2} \le \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1 - P_\mu\left\{-z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right\} \\ &= 1 - \Phi\left(z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right) + \Phi\left(-z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right)\end{aligned}$$

```
> zalpha = qnorm(.975)
> mu=(-200:200)/100
> pi = 1 - pnorm(zalpha-5*mu)+pnorm(-zalpha-5*mu)
> plot(mu,pi,type="l")
```

2 The p-value

The report of *reject* the null hypothesis does not describe the strength of the evidence because it fails to give us the sense of whether or not a small change in the values in the data could have resulted in a different decision. Consequently, the common method is not to choose, in advance, a level α of the test and then report "reject" or "fail to reject", but rather to report the value of the test statistic and to give all the values for α that would lead to the rejection of H_0.

For example, if the test is based on having a test statistic $S(X)$ exceed a level k, i.e., we have decision function

$$d_k(X) = 1 \text{ if and only if } S(X) \geq k.$$

and if the value $S(X) = k_0$ is observed, then the p-value equals

$$\sup\{\pi_{d_t}(\theta); \theta \in \Theta_0\} = \sup\{P_\theta\{S(X) \geq k_0\}; \theta \in \Theta_0\}.$$

In the one-sided test above, if $\bar{X} = 1$, then

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{1-0}{1/\sqrt{25}} = 5.$$

```
> pvalue = 1 - pnorm(5)
> pvalue
[1] 2.866516e-07
```

In this case, the p-value is 2.87×10^{-7}.

3 Confidence Sets

Choose a number γ between 0 and 1. From data X, suppose that we compute two statistIcs $L(X)$ and $R(X)$ so that irrespective of the value of the parameter,

$$P_\theta\{L(X) < \theta < R(X)\} \geq \gamma.$$

If $\ell = L(X)$ and $r = R(X)$ are the observed values based on the data X, then the interval (ℓ, r) is called a **confidence interval** for θ with **confidence coefficient** γ. This notion extends the idea of a point estimator $\hat{\theta}$ by adding a notion concerning how closely we can estimate θ.

We will show how an α test for the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

generates a $\gamma = 1 - \alpha$ confidence interval.

Let d denote the decision function for this test. Given the data X, let $\omega(X)$ denote those parameter values for which the test fails to reject this hypothesis. Thus,

$$\theta_0 \in \omega(X) \text{ if and only if } d(X) = 0.$$

and

$$P_{\theta_0}\{\theta_0 \in \omega(X)\} = P_{\theta_0}\{d(X) = 0\} = 1 - P_{\theta_0}\{d(X) = 1\} = 1 - \alpha = \gamma.$$

Now let $L(X)$ and $R(X)$ be the end points of the interval $\omega(X)$.

Example 3. *In the example above, for the two-sided test based on normal data*

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0$$

$$d(X) = 0 \text{ if and only if } \left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| < z_{\alpha/2}.$$

$$\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \mu_0 \leq \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\bar{X} \quad \text{and} \quad \omega(X) = \left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right).$$