Likelihood Ratio Tests

April 3, 2008

The likelihood ratio test is a popular choice a composite hypothesis.

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

when Θ is a multidimensional parameter space and Θ_0 is a subspace.

$$\Lambda(\mathbf{x}) = \frac{\sup\{L(\theta|\mathbf{x}); \theta \in \Theta_0\}}{\sup\{L(\theta|\mathbf{x}); \theta \in \Theta\}}$$

The rejection region for an α-level test is $\{\Lambda(\mathbf{x}) \leq \lambda_0\}$ where λ_0 is chosen so that

$$P_\theta\{\Lambda(X) \leq \lambda_0\} \leq \alpha \text{ for all } \theta \in \Theta_0.$$

Let $\hat{\theta}_0$ be the parameter value that maximizes the likelihood for $\theta_0 \in \Theta_0$ and $\hat{\theta}$ be the parameter value that maximizes the likelihood for $\theta_0 \in \Theta$. Then,

$$\Lambda(\mathbf{x}) = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}.$$

Example 1. *Let $\Theta = \mathbb{R}$ and consider the* **two-sided hypothesis**

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

Here the data are n independent $N(\mu, \sigma^2)$ random variables $X_1 \ldots, X_n$ with known variance σ^2. Then, $\hat{\mu}_0 = \mu_0$ and $\hat{\mu} = \bar{x}$. Consequently,

$$L(\hat{\mu}_0|\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^n \exp -\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu_0)^2, \quad L(\hat{\mu}|\mathbf{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^n \exp -\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x})^2$$

and

$$\Lambda(\mathbf{x}) = \exp -\frac{1}{2\sigma^2}\left(\sum_{i=1}^n ((x_i - \mu_0)^2 - (x_i - \bar{x})^2)\right) = \exp -\frac{n}{2\sigma^2}(\bar{x} - \mu_0)^2.$$

Now notice that

$$-2 \ln \Lambda(\mathbf{x}) = \frac{n}{\sigma^2}(\bar{x} - \mu_0)^2 = \left(\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right)^2.$$

Because $(\bar{X} - \mu_0)/(\sigma/\sqrt{n})$ is a standard normal random variable, $-2 \ln \Lambda(X)$ is the square of a standard normal, hence, a χ-square random variable with 1 degree of freedom.

1 Chi-square test

This exact computation for normal data yields, owing to the central limit theorem, an asymptotic result that is contained in the following theorem.

Theorem 2. *Whenever the maximum likelihood estimate has an asymptotically normal distribution, let* $\Lambda_n(\mathbf{x})$ *be the likelihood ratio criterion for*

$$H_0 : \theta_1 = c_1 \text{ for all } i = 1, \dots, k \quad \text{versus} \quad H_1 : \theta_1 \neq c_1 \text{ for some } i = 1, \dots, n$$

Then under H_0*,*

$$-2 \ln \Lambda_n(X)$$

converges in distribution to a χ^2_k *random variable.*

Example 3. *Let* $X_1 \dots X_n$ *be independent* $Pois(\lambda)$ *random variables and consider the test*

$$H_0 : \lambda = \lambda_0 \quad \text{versus} \quad H_1 : \lambda \neq \lambda_0.$$

Then the likelihood

$$L_n(\theta|\mathbf{x}) = \frac{\lambda^{x_1}}{x_1!} e^{-\lambda} \cdots \frac{\lambda^{x_n}}{x_n!} e^{-\lambda} = \frac{1}{x_1! \cdots x_n!} \lambda^{x_1 + \cdots + x_n} e^{-n\lambda}$$

The maximum likelihood is taken for $\lambda = \bar{x}$*. Thus,*

$$\Lambda(\mathbf{x}) = \frac{\lambda_0^{x_1 + \cdots + x_n} e^{-n\lambda_0}}{\bar{x}^{x_1 + \cdots + x_n} e^{-n\bar{x}}} = \frac{\bar{x}^{n\bar{x}} e^{-n\bar{x}}}{\lambda_0^{n\bar{x}} e^{-n\lambda_0}}$$

and

$$-2 \ln \Lambda_n(X) = -2n(\bar{x} \ln \lambda_0 - \lambda_0 - \bar{x} \ln \bar{x} + \bar{x}).$$

To determine the critical values for this test, we have

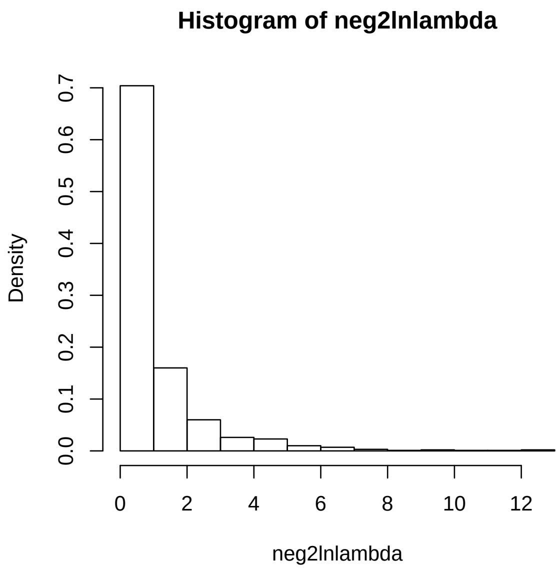
```
> qchisq(c(0.90,0.95,0.98,0.99),1)
[1] 2.705543 3.841459 5.411894 6.634897
```

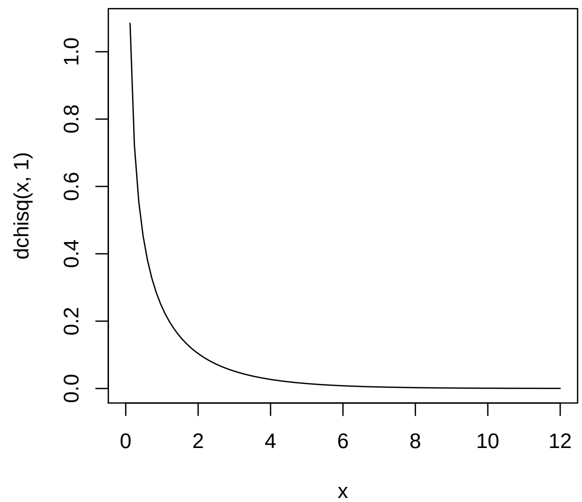
We compare $-2 \ln \Lambda_n(X)$ *to the* χ^2_1 *distribution with* $n = 36$ *and* $\lambda = 3$ *using 1000 simulations under the null hypothesis* $\lambda = 3$*.*

```
> neg2lnlambda <-rep(0,1000)
> n=36
> lambda=3
> for(i in 1:1000){x<-rpois(n,3);
neg2lnlambda[i] =-2*n*(mean(x)*log(lambda)-lambda-mean(x)*log(mean(x))+mean(x))}
> hist(neg2lnlambda,probability=TRUE)
```

and

```
> curve(dchisq(x,1),0,12)
```


Histogram of neg2lnlambda
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
2
4
6
8
10
12
neg2lnlambda


dchisq(x, 1)
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
x