Math 466 - Practice Problems for Exam I

1. A random variable X is uniformly distributed between 0 and 1 if the pdf is $f_X(x) = 1$ for $0 \leq x \leq 1$. The moment generating function of such a random variable is

$$M_X(t) = \frac{e^t - 1}{t} \tag{1}$$

You can use this fact without deriving it.

(a) Find the mean and variance of a random variable X that is uniformly distributed between 0 and 1.
(b) Let $X_1, X_2, \cdots, X_n$ be independent random variables each of which is uniformly distributed between 0 and 1. Let

$$\bar{X_n} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{2}$$

be the usual sample mean. Find the mean and variance of $\bar{X_n}$.
(c) For large n

$$P(0.5 \leq \bar{X_n} \leq 0.51) \approx \frac{1}{\sqrt{2\pi}} \int_0^{\alpha_n} e^{-z^2/2} \, dz \tag{3}$$

Find α_n. It should depend on n. (d) Find the moment generating function of $Y = \sum_{i=1}^{n} X_i$.

2. The moment generating function of a random variable X is

$$M(t) = \left(\frac{1}{1-t} \right)^4$$

(a) Find the mean and variance of X.

3. A random sample $\{X_1, X_2, X_3\}$ of size 3 is drawn from a population with mean μ and variance σ^2. (So X_1, X_2, X_3 are i.i.d.) Let

$$T_1 = \frac{1}{3}(X_1 + X_2 + X_3)$$
$$T_2 = \frac{1}{4}(X_1 + 2X_2 + X_3)$$

(a) Show that T_1 and T_2 are both unbiased estimators of the population mean μ.

(b) Compute the variances of T_1 and T_2.

(c) Which estimator would you use? Explain your answer.

4. Consider the following density

$$f(x|\theta) = \theta^2 \, x \, e^{-\theta x}, \quad x \geq 0; \qquad f(x|\theta) = 0, \quad x < 0$$

It is easy to show the integral of this is 1, the mean is $\mu = 2/\theta$ and the variance is $\sigma^2 = 2/\theta^2$. (You can assume this.)
(a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.
(b) Find the maximum likelihood estimator $\hat{\mu}$ of the mean μ.

5. Let

$$f(x|\theta) = \frac{1}{2}\theta^3 \, x^2 \, e^{-\theta x}, \quad x > 0$$

Some calculus shows that the mean of this distribution is $\mu = 3/\theta$ and the variance is $\sigma^2 = 3/\theta^2$. (You may assume this without showing it.)
(a) For a random sample of size n, let $\overline{X_n}$ be the sample mean. Find its mean and variance.
(b) The sample mean is an unbiased estimator of μ. Show that no other unbiased estimator is better in the sense that no other unbiased estimator has smaller variance.

6. Let

$$f(x|\theta) = c\theta \exp(-\theta^4 x^4), \quad -\infty < x < \infty$$

where c is the constant that makes this a probability density, i.e.,

$$c^{-1} = \int_{-\infty}^{\infty} \exp(-x^4) \, dx$$

(a) Find the maximum likelihood estimator of θ.
(b) Show that the variance σ^2 equals $a\theta^{-2}$ where

$$a = c \int_{-\infty}^{\infty} x^2 \exp(-x^4) \, dx$$

(c) Find the maximum likelihood estimator of σ^2.

7. The population has a uniform distribution on $[0, \theta]$ with θ unknown. So

$$f(x|\theta) = \frac{1}{\theta} \text{ if } 0 \leq x \leq \theta$$

We take a Bayesian point of view, and assume the prior distribution of θ is

$$\pi(\theta) = \begin{cases} 3\theta^2 & \text{if } 0 \leq \theta \leq 1 \\ 0 & \text{if } \theta \notin [0, 1] \end{cases}$$

We consider a random sample of size $n = 2$.
(a) What is the density of the random sample given θ, i.e., $f(x_1, x_2|\theta)$?
(b) What is the joint density of the random sample and θ, i.e., $f(x_1, x_2, \theta)$?
(c) Show that the posterior density of θ, $\pi(\theta|x_1, x_2)$, is uniform on some interval and determine the interval. (It should depend on x_1, x_2.)
(d) If we use squared error loss, what is the Bayes estimator of θ ? If you couldn't do part (c), let $[a, b]$ be the answer to (c) and give the estimator in terms of a and b.

8. The population has a uniform distribution on $[-\theta, \theta]$ with θ unknown. So

$$f(x|\theta) = \frac{1}{2\theta} \text{ if } -\theta \leq x \leq \theta$$

We take a Bayesian point of view, and assume the prior distribution of θ is

$$\pi(\theta) = \begin{cases} 4\theta^3 & \text{if } 0 \leq \theta \leq 1 \\ 0 & \text{if } \theta \notin [0, 1] \end{cases}$$

We consider a random sample of size $n = 3$.
(a) What is the density of the random sample given θ, i.e., $f(x_1, x_2, x_3|\theta)$?
(b) Show that the posterior density of θ, $\pi(\theta|x_1, x_2, x_3)$, is uniform on some interval and determine the interval. (It should depend on x_1, x_2, x_3.)
(c) If we use squared error loss, what is the Bayes estimator of θ ?

9. A factory produces widgets which can be either good or defective. Let p be the proportion of the entire population of widgets that are defective. We believe that the proportion of defective widgets is at most 0.5. We use a Bayesian approach and take the prior distribution of p to be the uniform distribution on $[0, 0.5]$. In a sample of n widgets, all of them are defective.
(a) Find the posterior distribution of p for this particular sample.

(b) If we use squared error loss, what is the Bayesian estimator for p for this particular sample?

10. The waiting time for phone support at Microsludge has an exponential distribution with parameter θ and hence mean $\mu = 1/\theta$. The sample mean $\overline{X_n}$ is an estimator for the mean μ. We consider estimators of the form $a\overline{X_n}$.
(a) Compute the variance of the estimator $a\overline{X_n}$.
(b) Compute the bias of the estimator $a\overline{X_n}$.
(c) We use the squared error loss function. Compute the risk of $a\overline{X_n}$.
(d) Find the value of a that minimizes the risk.