Uniformly Most Powerful Tests

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We begin with a composite hypothesis test

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

with $\Theta_0 \cap \Theta_1 = \emptyset$ and $\Theta_0 \cup \Theta_1 = \Theta$. Let d be the decision function for an α level test. Thus, the power function

$$\pi_d(\theta) \leq \alpha \quad \text{for all } \theta \in \Theta_0.$$

Definition 1. *A test procedure d is a* **uniformly most powerful (UMP) test** *at the significance level α if d is indeed an α level test and if for any other α level procedure d^*,*

$$\pi_{d^*}(\theta) \leq \pi_d(\theta) \text{ for every value of } \theta \in \Theta_1.$$

In general, a hypothesis will not have a uniformly most powerful test. However, in several procedures involving simple hypothesis, the test statistic did not depend on the specific value of the alternative. For example, in the case of independent normal data with unknown mean μ and known variance σ^2, we have simple hypothesis

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu = \mu_1$$

with $\mu_1 > \mu_0$. The critical region is determined by the likelihood ratio test is

$$C = \{\mathbf{x}; \bar{x} \geq k_\alpha\}.$$

irrespective of the value of μ_1.

1 Monotone Likelihood Ratios

The key to obtaining this outcome is given in the following definition.

Definition 2. *Let $\mathbf{f}(\mathbf{x}|\theta)$ be the joint density function of the observations $X = (X_1 \ldots, X_n)$. Then $\mathbf{f}(\mathbf{x}|\theta)$ is said to have a* **monotone likelihood ratio in the statistic** *$T(X)$ if for any choice $\theta_1 < \theta_2$ of parameter values, the likelihood ratio*

$$\frac{\mathbf{f}(\mathbf{x}|\theta_2)}{\mathbf{f}(\mathbf{x}|\theta_1)}$$

depends on value of the data $\mathbf{x}$ only through the value of statistic $T(\mathbf{x})$ and, in addition, this ratio is a monotone function of $T(\mathbf{x})$.

Thus, for independent normal data with unknown mean μ and known variance σ^2, we have a increasing monotone likelihood ratio in the statistic $T(\mathbf{x}) = \bar{x}$.

Lemma 3. *Suppose that $\mathbf{f}(\mathbf{x}|\theta)$ has an increasing monotone likelihood ratio in the statistic $T(X)$ and let ψ be a nondecreasing function then,*

$$g(\theta) = E_\theta[\psi(T(X))]$$

is a non-decreasing function of θ

Proof. Let $\theta_1 < \theta_2$ and define

$$A = \{\mathbf{x}; \mathbf{f}(\mathbf{x}|\theta_1) > \mathbf{f}(\mathbf{x}|\theta_2)\}, \quad a = \sup_{\mathbf{x}\in A} \psi(T(x)),$$

and

$$B = \{\mathbf{x}; \mathbf{f}(\mathbf{x}|\theta_1) < \mathbf{f}(\mathbf{x}|\theta_2)\}, \quad b = \sup_{\mathbf{x}\in B} \psi(T(x)).$$

On A, the likelihood ratio is less than 1. On B, the likelihood ratio is greater than 1. Thus, because $\mathbf{f}(\mathbf{x}|\theta)$ has an increasing monotone likelihood ratio in the statistic $T(X)$ and ψ is a nondecreasing function, $b \geq a$. Note that

$$0 = \int (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx = \int_A (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx + \int_B (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx$$

and

$$\int_B (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx = -\int_A (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx.$$

Finally,

$$\begin{aligned} g(\theta_2) - g(\theta_1) &= \int \psi(T(\mathbf{x})(\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx \\ &\geq \int_A a(\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx + \int_B b(\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx \\ &= (b-a)\int_B (\mathbf{f}(\mathbf{x}|\theta_1) - \mathbf{f}(\mathbf{x}|\theta_2))\ dx \geq 0. \end{aligned}$$

□

Theorem 4. *Suppose that $\mathbf{f}(\mathbf{x}|\theta)$ has an increasing monotone likelihood ratio for the statistic $T(\mathbf{x})$. Let α and k_α be chosen so that*

$$\alpha = P_{\theta_0}\{T(X) \geq k_\alpha\}.$$

Then $C = \{\mathbf{x}; T(\mathbf{x}) \geq k_\alpha\}$ is the critical region for a uniformly most powerful α level test for the one-sided alternative hypothesis

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Proof. Let $d(\mathbf{x}) = I_C(\mathbf{x})$ be the decision function corresponding to the critical region C.

Use the previous lemma and the function $\psi(t) = I_{[k_\alpha,\infty)}(t)$ to see that the power function

$$\pi_d(\theta) = P_\theta\{d(X) = 1\} = P_{\theta_0}\{T(X) \geq k_\alpha\}$$

is a monotone increasing function of θ.

Thus, $\alpha = \pi_d(\theta_0) = \sup\{\pi_d(\theta); \theta \leq \theta_0\}$ and d is an α level test.

Pick $\theta_1 > \theta_0$ and consider the simple hypothesis

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1.$$

Then, by the Neyman-Pearson lemma, the best choice for critical region is to choose a number c_α so that

$$C = \left\{ \mathbf{x}; \frac{\mathbf{f}(\mathbf{x}|\theta_1)}{\mathbf{f}(\mathbf{x}|\theta_0)} \geq c_\alpha \right\}.$$

Because $\mathbf{f}(\mathbf{x}|\theta)$ has an increasing monotone likelihood ratio for the statistic $T(\mathbf{x})$, this is equivalent to

$$C = \{\mathbf{x}; T(\mathbf{x}) \geq k_\alpha\}$$

for some k_α. □

2 Examples

Example 5 (Cauchy random variables)**.** *Let X be a Cauchy random variable $Cau(\theta, 1)$. Then*

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\frac{1}{\pi}\frac{1}{1+(x-\theta_2)^2}}{\frac{1}{\pi}\frac{1}{1+(x-\theta_1)^2}} = \frac{1 + (x - \theta_1)^2}{1 + (x - \theta_2)^2}$$

This lilelihood ratio is not monotone in x.

Example 6 (Normal random variables)**.** *Let $X_1, \ldots, X_n$ be independent $N(0, \sigma^2)$ random variables Then*

$$\frac{\mathbf{f}(\mathbf{x}|\sigma_2^2)}{\mathbf{f}(\mathbf{x}|\sigma_1^2)} = \frac{(2\pi\sigma_2^2)^{-n/2} \exp -\frac{1}{2\sigma_2^2}\sum_{i=1}^n x_i^2}{(2\pi\sigma_1^2)^{-n/2} \exp -\frac{1}{2\sigma_1^2}\sum_{i=1}^n x_i^2} = \left(\frac{\sigma_1}{\sigma_2}\right)^n \exp - \left(\frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2}\right) \sum_{i=1}^n x_i^2.$$

Consequently, $\mathbf{f}(\mathbf{x}|\sigma^2)$ has an increasing monotone likelihood ratio in the statistic

$$T(X) = \sum_{i=1}^n X_i^2.$$

By the theorem above, $C = \{\mathbf{x}; \sum_{i=1}^n x_i^2 \geq k_\alpha\}$ is the critical region for a uniformly most powerful α level test for the one-sided alternative hypothesis

$$H_0 : \sigma^2 \leq \sigma_0^2 \quad \text{versus} \quad H_1 : \sigma^2 > \sigma_0^2.$$

Now,

$$\frac{1}{\sigma_0^2} \sum_{i=1}^n X_i^2$$

has a χ-square distribution with n degrees of freedom. So, if Y is a χ^2_n random variable and $\chi^2_{n,\alpha}$ satisfies

$$P\{Y > \chi^2_{n,\alpha}\} = \alpha,$$

then, $k_\alpha = \sigma_0^2 \chi^2_{n,\alpha}$.