Math 583B Spring 2012 Problem Set #3

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~~Exercises due Wednesday, 2/29~~

~~Problems~~ Everything is due Friday, 3/2

Exercises. *These do not need to be turned in.*

1. Show that the Sturm-Liouville operator defined in class is self-adjoint.

2. Consider this example from class:

$$x^2y''(x) + xy'(x) + y(x) = \mu y(x)\ . \tag{1}$$

   (a) By considering solutions of the form $y(x) = cx^\alpha$, find the general solution of this equation. *Hint: The general solution has the form $c_1f_1(x)+c_2f_2(x)$, where each of the f_i is has the form above. You will find that the solutions are complex, however. By taking linear combinations, you can express every solution in the form $c_1'g_1(x) + c_2'g_2(x)$ where the $g_i(x)$ are real.* **expanded 2/28**

   (b) Now suppose we want to solve the eigenvalue problem on the interval $1 \leq x \leq 2$ with boundary conditions $y(1) = y(2) = 0$. Find the eigenvalues using the general solution above. (Remember we showed in class that $\mu \leq 1$.)

3. Let H be the space of square-integrable functions on $[0, 1]$ such that $u(0) = 0$ and $u'(0) = u(1)$, and let L be the operator $\frac{d^2}{dx^2}$ acting on (a dense subset of) H . Recall that the adjoint L^* is characterized by

$$\langle Lu, v\rangle = \langle u, L^*v\rangle \tag{2}$$

   for all $u \in D(L)$ and all $v \in D(L^*)$, where $D(L)$ is the domain of L (in this case, we can take it to be the space of twice-differentiable functions in H), and the domain $D(L^*)$ of L^* is such that Eq. (2) holds. L is self-adjoint if $L = L^*$ (and $D(L) = D(L^*)$).

   (a) Is L self-adjoint?

   (b) Consider the boundary value problem $Lu = \lambda u$ for $u \in H$. Show that the eigenvalue λ must satisfy $\sqrt{\lambda} = \sin(\sqrt{\lambda})$. Are there any real eigenvalues?

   (This exercise illustrates a subtlety in the notion of self-adjoint operators, namely that when we say "$L = L^*$," we assume implicitly that the two operators are defined on the same domain. There is a bit more discussion on this example in Ch. 6 of the notes.)

Problems. *Please write these up and turn them in.*

1. This problem explores a phase-plane view of the Sturm-Liouville problem (see Sect. 6.3.2 of the course notes for a more general formulation, but there is an error in Fig. 6.1).

   Suppose $u : [0,1] \to \mathbb{R}$ is an eigenfunction for a regular Sturm-Liouville problem with Dirichlet boundary conditions $u(0) = u(1) = 0$. Let λ denote the corresponding eigenvalue, and let $v(x) = p(x)u'(x)$.

   (a) Using the function v above, we can rewrite the Sturm-Liouville equation as a pair of ODEs. Rewrite these ODEs in polar coordinates by letting $u(x) = r(x)\sin(\theta(x))$ and $v(x) = r(x)\cos(\theta(x))$ and deriving equations for θ and r .

   (b) What conditions on (r, θ) correspond to the boundary conditions?

   (c) Assuming q and $\sigma > 0$ are constants, show that $\theta'(x) > 0$ for all x . **assumption added 2/28**

   (d) Let's order the eigenvalues so that $\lambda_1 < \lambda_2 < \cdots$. Assuming θ' is uniformly positive, sketch what phase plane curves corresponding to ϕ_1 , ϕ_2 , and ϕ_3 *should* look like. Also sketch the corresponding graphs as functions of x.

   (e) How many zeros does ϕ_n have inside $(0, 1)$? How are the zeros of ϕ_n related to those of ϕ_{n+1}?

   Note: This should give you a hint as to why the eigenvalues do not accumulate except at ∞.

2. For the inhomogeneous equation

$$x^2y''(x) + xy'(x) + y(x) = f(x) \ , \quad 1 \le x \le 2 \ , \quad y(1) = y(2) = 0 \ ,$$

   write down a general expression for the solution as an orthogonal expansion using the eigenfunctions of Eq. (1).