Math 583B Spring 2012 Problem Set #4

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Exercises Due Wednesday, 3/21
Problems Due Friday, 3/23

Exercises. *These do not need to be turned in.*

1. Consider the equation
$$u'' + u = f\ ,\ \ 0 \le x \le \ell\ ,$$
where f is an arbitrary given function and u satisfies periodic boundary conditions. For what values of ℓ is this equation *well-posed*, i.e., has a unique solution for every $f \in L^2([0,\ell])$?

2. Physically, square-integrable (L^2) solutions of the time-independent Schrödinger equation
$$-\frac{1}{2}\psi''(x) + V(x)\psi(x) = E\psi(x)$$
represent "bound states," i.e., they are probability distributions in situations where the particle cannot escape to infinity. Unbound states, characterized by wave functions *not* in L^2, do occur in QM: they represent particles that can escape to infinity; such states (and associated spectral problems) are important in the study of scattering.

   Show that the free particle equation (i.e., with $V \equiv 0$) has (i) smooth eigenfunctions, but (ii) no bound states.

3. Consider the harmonic oscillator in Eq. (1). Using the variational method and the trial functions $\varphi_a(x) = e^{-ax^2}$, find the optimal number a_* and the corresponding energy $E(a_*)$, where $E(a) = \langle H\varphi_a, \varphi_a\rangle \,/\, \langle \varphi_a, \varphi_a\rangle$. (I asserted the results without showing you the work in class.)

Problems. *Please write these up and turn them in.*

1. Consider again the harmonic oscillator in Eq. (1). Estimate the ground state energy using the variational method and the trial functions
$$\varphi_a(x) = \begin{cases} (x-a)^2(x+a)^2\ , & |x| \le a \\ 0, & |x| > a \end{cases}$$
How close do you get to the right answer?

2. Consider the time-independent Schrödinger equation for the harmonic oscillator
$$-\frac{1}{2}\psi''(x) + \frac{x^2}{2}\psi(x) = E\psi(x)\ . \tag{1}$$

In this problem we will determine the eigenfunctions and eigenvalues.[1]

(a) Heuristically, when $|x| \gg 1$, the x^2 term in Eq. (1) will dominate the $E\psi$ term, so that for large $|x|$ we have $-\frac{1}{2}\psi'' + \frac{x^2}{2}\psi \approx 0$. This implies, among other things, that all eigenfunctions behave in essentially the same way as $|x| \to \infty$. Since we "know" (or can at least guess, e.g., by using the variational method) that the ground state is $\psi_0(x) = e^{-x^2/2}$, it is natural to represent all other eigenfunctions as multiples of the ground state, i.e., define u by $\psi(x) = u(x)\psi_0(x)$.

Derive the eigenvalue equation for u.

(b) Let $u(x) = \sum_{n=0}^{\infty} c_n x^n$. Using the differential equation in (a), find a recurrence relation for c_{n+2} in terms of c_n . (Don't worry about the convergence of the power series.) Note that the recurrence relation will involve the (still unknown) eigenvalue E .

(c) Show that if $E = n+\frac{1}{2}$, where n is a nonnegative integer, then the corresponding eigenfunctions u are polynomials.[2] Thus for the original problem, $\psi(x) = u(x)\psi_0(x)$ will decay like a gaussian as $x \to \pm\infty$, and will be in $L^2(\mathbb{R})$.

(d) Suppoe u is not a polynomial, i.e., its Taylor series is truly infinite. Argue that $c_{n+2}/c_n \approx 2/n$ for n large. What does this suggest about the behavior of $u(x)$ for large x, and that of $\psi(x)$?

[1] You will not be asked to write down the eigenfunctions explicitly, but it should become clear that one can find any particular eigenfunction you want using this method. Also, in quantum physics courses, it is customary to solve this problem via so-called "raising" and "lowering" operators; see the Wikipedia page on the "quantum harmonic oscillator."

[2] These polynomials are known as *Hermite polynomials.* They form an orthogonal basis for the Hilbert space of functions with inner product $\langle f, g\rangle = \int_{\mathbb{R}} f(x)\, g(x)\, e^{-x^2}\, dx$.