# Math 583B Spring 2012 Problem Set #6

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Due Wednesday, 4/18

**Read** Sects. 7.1 - 7.4 of the course notes.

**Exercises.** *These do not need to be turned in.*

1. Let $L$ be a bounded operator on a Hilbert space $H$, and suppose $u, f \in H$ and $f \perp ker(L^*)$ . Show that $Lu = f$ if and only if $L^* L u = L^* f$ .[1]

2. Given an explicit solution if possible. If there is no solution, explain why. (You can assume $x$ ranges over the interval of integration.)

   (a)
   $$u(x) = 1 + \int_0^1 \sinh(x-y) u(y)\, dy$$

   (b)
   $$u(x) = 1 + \int_{-1}^1 e^{x^2+y^2} \sin(x) \sin(y) u(y)\, dy$$

**Problems.**

1. In this problem, we will derive the Fredholm alternative for 2nd-order boundary value problems of the form

$$\begin{aligned} p_2(x)u''(x) + p_1(x)u'(x) + p_0(x)u(x) = f(x)\ , \ \ 0 \le x \le 1 \\ \alpha_1 u + \alpha_2 u' = 0 \text{ at } x = 0 \\ \beta_1 u + \beta_2 u' = 0 \text{ at } x = 1 \end{aligned} \qquad (1)$$

   For this problem, assume $p_2 > 0$, and that there exists a nonzero function $u_0$ such that $p_2 u_0'' + p_1 u_0' + p_0 u_0 = 0$ and $u_0$ satisifies the boundary conditions. **clarified**

   (a) Let's first derive a solvability condition via a Sturm-Liouville reduction. Recall that for a Sturm-Liouville problem

   $$(pu')' + qu = g$$

   with the same boundary conditions as above, if there exists a nonzero $u_0$ such that $(pu_0')' + qu_0 = 0$, then the problem has solution if and only if $\langle u_0, g\rangle = 0$ . (We showed this earlier by eigenfunction expansions.) Find a solvability condition (in terms of $u_0$ and $f$) for the BVP in Eq. (1) by reduction to Sturm-Liouville form.

[1] When $L$ is a matrix, this is just the *normal equation* from linear algebra.

(b) Next, we need to know something about the adjoint problem. Let $Lu = p_2 u'' + p_1 u' + p_0 u$ acting on the space of functions satisfying the boundary conditions in Eq. (1). Note that there is no reason to think $L$ is self-adjoint; consequently, the adjoint $L^*$ may be defined on a space with boundary conditions different from $L$. (You may want to review Sect. 6.2.3 of the notes.) Show that $\underline{L^* v = (p_2 v)'' - (p_1 v)' + p_0 v}$ , and find the associated boundary conditions. **fixed sign**

(c) Show that the solvability condition you found earlier is equivalent to $f \perp ker(L^*)$ . *Hint: The solvability condition you found in (a) should have the form* $\int_0^1 f(x) v_0(x)\, dx = 0$ *for some function* $v_0$*. Show that* $L^* v_0 = 0$ *and that* $v_0$ *satisfies the boundary conditions for* $L^*$ *.*

2. Let $k(x,y) = 1 + \sin(\pi x)\cos(\pi y)$, and define $K : L^2([0,1]) \to L^2([0,1])$ by **updated $k$**

$$(Kf)(x) = \int_0^1 k(x,y)\, f(y)\, dy$$

(a) Find all the eigenfunctions and eigenvalues of $K$ , i.e., find all $\lambda$ and $f \in L^2$ such that $Kf = \lambda f$ .

(b) Consider the equation

$$u = \mu K u + f \ ,$$

where $\mu$ is a given number. Under what conditions on $\mu$ and $f$ does this equation have a solution? When is the solution unique?