Math 254-009 Note on existence and uniqueness

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Last revised: September 13, 2016

Existence and uniqueness for autonomous first-order ODEs. A number of you asked me about the Existence and Uniqueness Theorem (EUT) in office hours yesterday. This note is an attempt to clarify some of the issues. The version of the EUT discussed here is less general than what's in the text, but also hopefully easier to understand.

My aim is to

- give examples of how to "use" the EUT on concrete examples, and
- give examples showing why the theorem says what it says.

I am not going to try to prove this theorem. If you are interested, Group Project B in Chapter 1 of the text outlines a standard proof.

Here is the EUT for first-order autonomous equations, which is the only one we will need.

> **Theorem.** Let $f(y)$ be a function that is continuous for $a < y < b$, where $a < b$ defines an interval in the real line. Furthermore, suppose $f'(y)$ is also continuous on the same interval. Then for all y_0 such that $a < y_0 < b$, the initial value problem
>
> $$\frac{dy}{dx} = f(y)\ , \quad y(x_0) = y_0 \tag{1}$$
>
> has a unique solution $y(x)$ defined for all x in an interval around x_0.

What it says is that if $f(y)$ and $f'(y)$ are both continuous on an interval around y_0, then there exists a solution $y(x)$ to the IVP, defined for x in an interval around x_0.

Reading carefully (like a lawyer or mathematician), you will notice that it does *not* say that if f or f' were discontinuous at y_0, then there are no solutions, nor that the solutions would not be unique. Indeed, there are examples where f or f' is discontinuous, and one can still construct unique solutions to an IVP. But there are also situations where bad things happen. We have seen some examples in class; I'll add to that list below.

Ex. 1. Consider the IVP

$$\frac{dy}{dx} = \sqrt{y}\ , \quad y(0) = 1. \tag{2}$$

Does it have a unique solution? *Answer: yes, because* $\sqrt{y}$ *and its derivative* $-1/(2\sqrt{y})$ *are both continuous at* $y = 1$*, the EUT tells us there is a unique solution defined on an interval around* $x = 0$*.*

Ex. 2. Consider the same differential equation, but now with $y(1) = 0$. Does it have a unique solution? *Answer: not necessarily, because* $-1/(2\sqrt{y})$ *is discontinuous at* $y = 0$*, and the EUT does not apply.*

Indeed, in class we saw that in this situation, both $y(x) = 0$ and $y(x) = (x/2-1)^2$ are solutions in this case, so uniqueness of solutions fails.

Note: the material above this line was covered in lecture and on the homework, and so you are responsible for this on the exam. The material below is to clarify the EUT; it won't be on the exam.

What can go wrong if f itself were discontinuous? Here are two examples to think about. In the first one, we can still construct unique solutions, but in the second one it is just not possible.

Ex. 3. Let

$$f(y) = \begin{cases} 1, & y \leqslant 1 \\ 3, & y > 1 \end{cases} \tag{3}$$

and consider

$$y' = f(y)\ , \quad y(0) = 0. \tag{4}$$

Since $y(0) = 0$, and f is continuous on an interval around 0, the EUT applies and the IVP has a unique solution. In fact, since the equation just says $y' = 1$ as long as $y < 1$, we have $y(x) = x$.

But at $x = 1$, we have $y(1) = 1$, i.e., it hits the place where $f(y)$ is discontinuous. What to do? Clearly, if the solution $y(x)$ is still defined for $x > 1$, we would need to have $y'(x) = f(y(x)) = 3$ (because $y(x) > 1$ for $x > 1$; try graphing $y(x)$ vs x). One thing we can at $x = 1$ is to solve a new IVP

$$y' = 3\ , \quad y(1) = 1. \tag{5}$$

The solution is $y(x) = 3(x-1)+1$. We can combine this solution with our earlier solution to obtain

$$y(x) = \begin{cases} x, & x \leqslant 1 \\ 1+3(x-1), & x > 1 \end{cases} \tag{6}$$

You can check that this function is continuous, satisfies $y'(x) = 1$ whenever $y(x) < 1$, and $y'(x) = 3$ whenever $y(x) > 1$, and $y(0) = 0$. So even though it is not differentiable for all x, and is therefore not strictly speaking a solution of the ODE $y' = f(y)$, it comes pretty close. And for all practical purposes, we can call this a solution of the IVP (4), so long as you don't mind $y(x)$ being non-differentiable at one point.

Even if we had started exactly at the discontinuity, i.e., $y(0) = 1$, we can still make a sensible solution: just let $y(x) = 1+3x$. The only ambiguity here is at the initial moment, $x = 0$, because $f(y)$ is discontinuous at $y = 1$. But $f(y) > 0$ for both $y < 1$ and $y > 1$, so either way the solution moves to the right, and for all $x > 0$ there is no problem.

It might help to imagine applying Euler's method to this problem with a very small timestep $h > 0$. If $y(0) = 1$, we only use $f(y) = 1$ at the initial step; thereafter we get $f(y) = 3$.

But really bad things can also happen.

Ex. 4. Let

$$f(y) = \begin{cases} 1, & y \leqslant 1 \\ -1, & y > 1 \end{cases} \tag{7}$$

and consider

$$y' = f(y)\ , \quad y(0) = 0. \tag{8}$$

We can try to do the same thing again, but this time we have a problem: the solution for $y(0) > 1$ is given by $y(x) = y(0) - x$. That is, the solution one obtains for $y(0) > 1$ wants to move back to the discontinuity. If you think about this for a while, you can convince yourself that for our IVP, there is no way to continue the solution past $x = 1$. Moreover, if we had started with $y(0) = 1$, then there is no way to construct a solution for any $x > 0$.

Again, if you view this through the lens of Euler's method, you would see that the approximate Euler solutions will jump back and forth right around the discontinuity in this case.