Invariant Densities on Maps With Noise

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1 Introduction

An invariant density for a map is one for which the density, under the action of the map, remains unchanged. That is, given a probability density for the location of a particle in the state space at time n, we say that the density is invariant if it is also the probability density for the location of the particle at time $n+1$. Such densities provide a description of the long-time statistical behavior of a system and, therefore, are of particular interest when the underlying dynamics are chaotic. While a deterministic map may support multiple invariant densities, special attention is given to stochastically stable invariant densities. These *physical invariant measures* (see [1]) are characterized by their stability when the corresponding deterministic map is perturbed at each step by a small amount of random noise. More precisely, suppose ρ_0 is such a density for the deterministic map

$$x_{i+1} = f(x_i). \tag{1}$$

Then consider the "noisy map"

$$x_{i+1} = f(x_i) + \epsilon\eta_i, \tag{2}$$

where η_i are all independent, identically distributed random variables. If ρ_ϵ is an invariant denstiy of the "noisy map" then the stability of ρ_0 implies that $||\rho_\epsilon - \rho_0||_1 \to 0$ as $\epsilon \to 0$. Such stable invariant densities are of interest because they represent the long-term behavior one might expect to observe in a physical system. That is, even if the dynamics of a process are understood completely, it can be expected that any physical instance of such a system will be subject to some small amount of random and unaccounted for noise.

While for some special maps an analytic expression for such invariant densities can be derived, this is not the case in general. As such, even for simple maps on a circle, we can often only hope to compute a numerical approximation to an invariant density. Some of the techniques for accomplishing this have been studied in detail (see [1]) and are discussed below.

One question which may be asked in studying stable invariant measures is at what rate we can expect that $||\rho_\epsilon - \rho_0||_1 \to 0$ as $\epsilon \to 0$. In [2] it is demonstrated that, for maps with exponential decay of correlations, this rate, as a power of ϵ, can be predicted from ρ_0 with a simple heuristic. In the same work, it is shown that this heuristic fails to make accurate predictions when examining maps displaying "intermittency". After developing the framework for the numerical computation of invariant densities, some of these results will be discussed and recreated below.

2 Transfer Operators

A useful notion in understanding invariant densities is that of the transfer operator. Consider a map given by $f(x)$ on a manifold X. The transer operator T_f will be the operator acting on probability densities $\rho(x)$, on X, such that for all $A \subseteq X$ we have that

$$\int_A (T_f\rho)(x)dx = \int_{f^{-1}(A)} \rho(x)dx. \tag{3}$$

From this definition it can be seen that the transfer operator effectively pushes densities forward in time. That is, given a probability density $\rho(x)$, for the state of a dynamical system at time n, $(T_f\rho)(x)$ will be the density at time $n+1$. Notice that in (3), if we perform a change of variables it can be seen that

$$(T_f\rho)(x) = \sum_{y\in f^{-1}(x)} \frac{\rho(y)}{|det(Df(y))|}. \tag{4}$$

Given this definition, an invariant density for the map $f(x)$ is an eigenfunction of its associated transfer operator, with an eigenvalue of one. The task of computing invariant densities can then be reduced to forming an approximation of the transfer operator and finding such eigenfunctions.

The above defines transfer operators for deterministic maps. When noise is added as in (2), the effect is to take a stable invariant density and "spread" probability according to the distribution of η_i. This can be formalized by defining the operator

$$(G_\epsilon\rho)(x) = \frac{1}{\epsilon^d}\int_X g(\frac{x-y}{\epsilon})\rho(y)dy, \tag{5}$$

with $d = dim(X)$ and g being the density of the random variables η_i (see [2]). We can now define the noisy transfer operator

$$T_\epsilon = G_\epsilon T_f. \tag{6}$$

T_ϵ is then related to the noisy map (2) in the same fashion as T_f and the deterministic map (1). Similarly, if ρ_ϵ is an invariant density for the noisy map (2), it is then an eigenfunction of T_ϵ with an eigenvalue of one.

3 Numerics

One approach to computing invariant densities relies on discretizing the transfer operator according to the definition given by(4). In this case, a grid is placed on the space X and values of the new density, at gridpoints, are determined by calculating the sum. Notice that, for a grid point x, the points $y \in f^{-1}(x)$ are typically not a part of the grid and so $\rho(y)$ must be approximated by interpolation. This approach was used in the following example.

3.1 Deterministic Logistic Map

Consider the logistic map

$$f(x) = 4x(1-x). \tag{7}$$

In this case it can be shown that the map has the invariant density

$$\rho_0(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{8}$$

To numerically compute this density, a matrix approximation to the transfer operator was constructed according to the method described above. A uniform mesh of 10^3 points was used and values of densities between gridpoints were calculated using six-point polynomial interpolation.

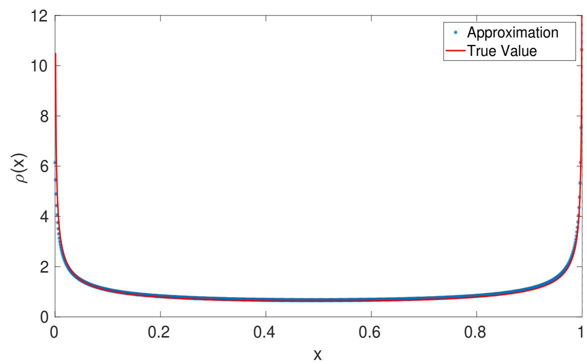

Figure 1: *The true value of ρ_0 and its approximation as given by the method described above.*

3.2 Ulam's Method

An alternative method for approximating the tranfser operator involves the construction of a stochastic matrix. The space X is first partitioned into N pieces $I_1, I_2, ..., I_N$. The $N \times N$ matrix $\hat{T}$ is then constructed with the entry in the ith column and jth row being given by

$$\hat{T}_{ij} = \frac{m(f^{-1}(I_i) \cap I_j)}{m(I_j)}, \tag{9}$$

where m is the Lebesgue measure. While this method has been seen to be less accurate [2] than the previously outlined approach, it has the advantage of producing an approximation of T, for which there is a natural interpretation. The entries of $\hat{T}$, as presented in (9), give the proportion of the set I_j which is moved to I_i by $f(x)$. So, the i, j entry of $\hat{T}$ may be interpreted as the probability of moving from states in I_j to states in I_i after one iteration of $f(x)$.

3.3 Intermittent Map With Noise

Consider the map

$$f(x) = \begin{cases} x + 2^{\alpha} x^{1+\alpha} & 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \frac{1}{2} < x < 1 \end{cases} \tag{10}$$

where $\alpha \in (0, 1)$. For this map it can be shown that the physical invariant density possesses an $x^{-\alpha}$ singularity at $x = 0$ [4]. Using Ulam's method with a uniform partition of 10^4 intervals, the invariant density was approximated.

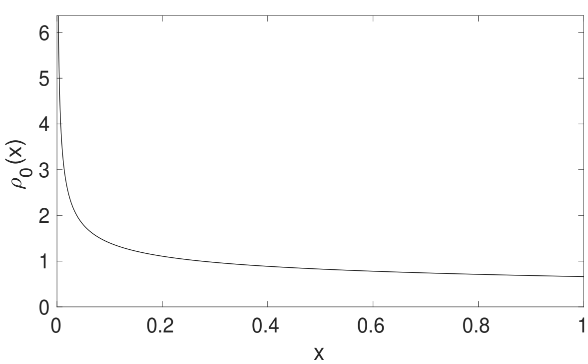

Figure 2: *The physical invariant density for the map with a parameter of* $\alpha = 0.5$. *The predicted singularity is observed in the approximation.*

The map with various levels of noise (2) was then considered, where all η_i were taken to have uniform distributions on the interval $[-1, 0]$. For the calculation of the integral in (5), densities were assumed to be constant on the intervals of the partition.

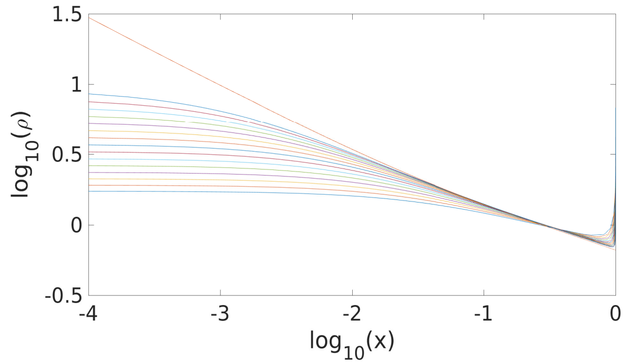

Figure 3: *The deterministic invariant density along with the invariant densities for levels of noise ranging from* $\epsilon = 10^{-1.5}$ *to* $\epsilon = 10^{-4}$.

Notice that the singularity at $x = 0$ is smoothed out with the introduction of only a small amount of noise.

4 Intermittency and the Convergence of Invariant Densities in the Small Noise Limit

Notice that (10) is tangent to the identity map at $x = 0$. As a result, a small positive value of x will take several iterations to escape from the region near zero. Once away from this region, the system behaves chaotically before eventually returning to a point near zero. This pattern of periods of regular behavior interrupted by bursts of chaos is what characterizes intermittency. This behavior can be observed in a number of physical systems (see [5]).

In [2] it is shown empirically that $||\rho_\epsilon - \rho_0||_1 \sim ||G_\epsilon \rho_0 - \rho_0||_1$ for systems with an exponential decay of correlations, but for maps such as (10), which display intermittent behavior and lack such a decay in correlations, this doesn't hold. In particular, it is noted in [2] that this estimate would lead one to expect $||\rho_\epsilon - \rho_0||_1 \sim \epsilon^{1-\alpha}$, which upon numerical inspection turns out not to be the case. To see this, the convergence studies of [2] were replicated using Ulam's method as described above.

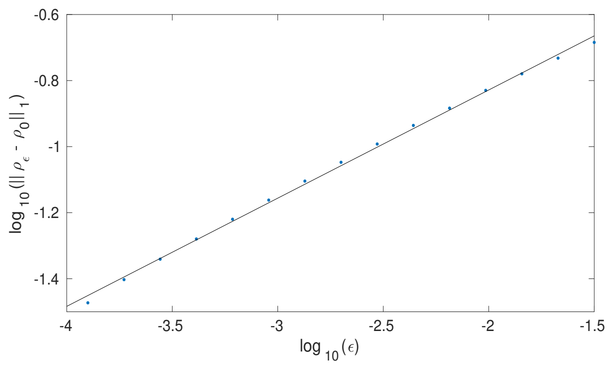

Figure 4: *The above comes from considering (10) with* $\alpha = 0.5$*. Numerical results indicate* $||\rho_\epsilon - \rho_0||_1 \sim \epsilon^\gamma$ *where* $\gamma = 0.3276 \pm .0084$*. This is in close agreement with the results of [2] and is much smaller than the estimate of* $\gamma = 1 - \alpha$*.*

These numerical experiments suggest that perhaps there exists an asymptotic estimate for γ, which accounts for the intermittent behavior of (10).

References

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