Lecture 12 notes

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Today we discussed

- The existence of stationary distributions (Theorem 1.24), with a partial proof, up to showing
$$\sum_{z\in S} \mu_x(z) p(z,y) = \mu_x(y). \tag{1}$$
Note some of the details were left out of the text and also the lecture; you will fill it in for Homework 6. (The rest of the proof is pretty clearly explained in the text.)
- The Perron-Frobenius Theorem.

Theorem 1.24. Here is a cleaned up proof of the following part of Theorem 1.24: suppose (X_n) is an irreducible chain where every state is recurrent. For $x, y \in S$, let

$$\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n). \tag{2}$$

We began by observing that

$$\sum_z \mu_x(z)\ p(z,y) = \sum_z \sum_{n=0}^{\infty} P_x(X_n = z, T_x > n)\ p(z,y) \tag{3a}$$
$$= \sum_{n=0}^{\infty} \sum_z P_x(X_n = z, T_x > n)\ P(X_{n+1} = y \mid X_n = z). \tag{3b}$$

If $n = 0$, then (as explained in class)

$$\sum_z P_x(X_0 = z, T_x > 0)\ P(X_{n+1} = y \mid X_n = z) = p(x,y). \tag{4}$$

If $n > 0$, then $P_x(X_n = z, T_x > n) = 0$ if $z = x$. So we have

$$\sum_z \mu_x(z) p(z,y) = p(x,y) + \sum_{n=1}^{\infty} \sum_{z\neq x} P_x(T_x > n \mid X_n = z)\ P_x(X_{n+1} = y \mid X_n = z)\ P_x(X_n = z). \tag{5}$$

Recall now that for a Markov chain, the future and the past are conditionally independent given the present. (This is on Homework 6.) This and $z \neq x$ imply

$$\begin{aligned} &P_x(T_x > n \mid X_n = z)\; P_x(X_{n+1} = y \mid X_n = z)\; P_x(X_n = z) \\ &= P_x(T_x > n, X_{n+1} = y, X_n = z). \end{aligned}$$

Up to this point, all was as discussed in class.

We now observe that if $y \neq x$, then

$$P_x(T_x > n, X_{n+1} = y, X_n = z) = P_x(T_x > n+1, X_{n+1} = y, X_n = z) \tag{6}$$

because knowing $X_{n+1} = y$ and $x \neq y$, we must have $T_x \neq n+1$, so that $T_x > n+1$ (since $T_x > n$ to start with). So

$$\begin{aligned} \sum_z \mu_x(z)\; p(z,y) &= p(x,y) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x > n, X_{n+1} = y, X_n = z) && \text{(7a)} \\ &= p(x,y) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x > n+1, X_{n+1} = y, X_n = z) && \text{(7b)} \\ &= \sum_{n=0}^{\infty} \sum_{z} P_x(T_x > n+1, X_{n+1} = y, X_n = z) && \text{(7c)} \\ &= \sum_{n=0}^{\infty} P_x(T_x > n+1, X_{n+1} = y). && \text{(7d)} \end{aligned}$$

From the second to the third line, we used similar reasoning as above (but in reverse). In the last step, we used that X_n had to be something. But this is just $\mu_x(y)$, reindexed. So

$$\sum_z \mu_x(z)\; p(z,y) = \mu_x(y) \tag{8}$$

if $y \neq x$.

On the other hand, if $x = y$, then

$$\begin{aligned} P_x(T_x > n, X_{n+1} = y, X_n = z) &= P_x(T_x > n, X_{n+1} = x, X_n = z) && \text{(9a)} \\ &= P_x(T_x = n+1, X_n = z) && \text{(9b)} \end{aligned}$$

because if $X_n = z$, $X_{n+1} = x$, and $x \neq z$, then we must have $T_x = n+1$ (which subsumes $T_x > n$). So we have

$$\begin{aligned} \sum_z \mu_x(z)\; p(z,x) &= p(x,x) + \sum_{n=1}^{\infty} \sum_{z \neq x} P_x(T_x = n+1, X_n = z) && \text{(10a)} \\ &= p(x,x) + \sum_{n=1}^{\infty} P_x(T_x = n+1) && \text{(10b)} \\ &= \sum_{n=0}^{\infty} P_x(T_x = n+1) && \text{(10c)} \end{aligned}$$

the last line because $P_x(T_x = 1) = p(x,x)$. By recurrence, $P_x(T_x < \infty) = 1$, so the above sums to 1. But one can check that

$$\mu_x(x) = \sum_{n=0}^{\infty} P_x(X_n = x, T_x > n) \tag{11a}$$
$$= P_x(X_0 = x, T_x > 0) \tag{11b}$$
$$= 1 \tag{11c}$$

because $P_x(X_n = x, T_x > n) = 0$ for $n > 0$. Thus

$$\sum_z \mu_x(z)\, p(z,y) = \mu_x(y). \tag{12}$$

QED

Perron-Frobenius Theorem: see Notes for Lecture 13 for more information.