Lecture 14 notes

klin@math.arizona.edu

March 3, 2020

Exit distributions

Today, we largely followed Sect. 1.9 of the text. There are many more nice examples there than I can possibly cover. Some comments:

1) In Theorem 1.28, the condition

$$P_x(V_A \wedge V_B < \infty) > 0 \tag{1}$$

is ~~necessary~~ sufficient to ensure that the system of equations

$$h(x) = \sum_{y \in S} p(x,y) h(y) \tag{2}$$

with $h(a) = 1$ for $a \in A$ and $h(b) = 0$ for $b \in B$ has a unique solution. Note there always exists one solution, namely $h(x) = P_x(V_A < V_B)$, but Eq. (2) may be underdeterminied, as we saw in the simple example in class. The above condition thus guarantees $h(x) = P_x(V_A < V_B)$.

2) The "heart" of the proof of Theorem 1.28, I think, is the formula

$$h(x) = E_x h(X_T) \tag{3}$$

where $T = V_A \wedge V_B$. If you found the closely related

$$h(x) = E_x h(X_{n \wedge T}) \tag{4}$$

mysterious, consider $n = 2$: first, as pointed out in the text, we have

$$h(x) = E_x h(X_1) \tag{5a}$$

$$= \sum_{y \in S} p(x,y) h(y). \tag{5b}$$

Let us split the sum up according to whether $X_1 \in C$ or $X_1 \notin C$, obtaining

$$h(x) = \sum_{y \in C} p(x,y)h(y) + \sum_{y \in A \cup B} p(x,y)h(y) \tag{6}$$

When $X_1 = y \in C$, we can expand $h(y)$ again using Eq. (2), so that

$$h(x) = \sum_{y \in C} p(x,y) \sum_{z \in S} p(y,z)h(z) + \sum_{y \in A \cup B} p(x,y)h(y) \tag{7a}$$

$$= \sum_{y \in C} \sum_{z \in S} p(x,y)p(y,z)h(z) + \sum_{y \in A \cup B} p(x,y)h(y) \tag{7b}$$

$$= \sum_{y \in C} \sum_{z \in S} P(X_0 = x, X_1 = y, X_2 = z)h(z) + \sum_{y \in A \cup B} P(X_0 = x, X_1 = y)h(y). \tag{7c}$$

But the above is exactly $E_x h(X_{2 \wedge T})$: the first term corresponds to when $X_1 \in C$, so that $T \geqslant 2$, and the second term corresponds to when $X_1 \notin C$, so that $T = 1$. We can summarize the corresponding moves of X_n as: go two steps if possible, but if $X_1 \in C$ already, then stop. This "stopped" Markov chain is exactly $X_{2 \wedge T}$, and the sum above gives $E_x h(X_{2 \wedge T})$. Repeating this argument yields Eq. (4), whose limit as $n \to \infty$ gives Eq. (3).

3) The statement that

$$P_x(V_A \wedge V_B < \infty) > 0 \tag{8}$$

implies

$$P_x(V_A \wedge V_B < \infty) = 1 \tag{9}$$

can be seen as follows: Eq. (8) implies that for each x, there is an integer $n(x)$ such that $P_x(X_{n(x)} \in A \cup B)$. Let $n_0 = \max\{n(x) | x \in C\}$, which is finite n_0 because C is finite. Then we are guaranteed that for each x, $P_x(V_T \leqslant n_0) > 0$, i.e., there is some chance of reaching A or B in $\leqslant n_0$ steps. Let $p_0 = \min\{P_x(V_A \wedge V_B \leqslant n_0) \mid x \in C\}$. Since C is finite, $p_0 > 0$. For $x \in C$ and $m > 0$, we have

$$P_x(V_A \wedge V_B \geqslant mn_0) \leqslant (1 - p_0)^m, \tag{10}$$

because for the event on the left to occur, we have to manage to go mn_0 steps without hitting either A or B, and in mn_0 steps we get at least m tries, each with a probability p_0 of succeeding. Letting $m \to \infty$, we get Eq. (9).